# Topological Hochschild homology of ring functors and exact categories 

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#### Abstract

In analogy with Hochschild-Mitchell homology for linear categories topological Hochschild and cyclic homology (THH and TC) are defined for ring functors on a category $\mathscr{C}$. Fundamental properties of $T H H$ and $T C$ are proven and some examples are analyzed. A special case of a ring functor on an exact category $\mathfrak{C}$ is treated separately, and is compared with algebraic K-theory via a Dennis-Bökstedt trace map. Calling THH and TC applied to these ring functors simply $T H H(\mathbb{C})$ and $T C(\mathbb{C})$, we get that the iteration of Waldhausen's $S$ construction yields spectra $\left\{T H H\left(S^{(n)} \mathbb{C}\right)\right\}$ and $\left\{T C\left(S^{(n)} \mathbb{C}\right)\right\}$, and the maps from K-theory become maps of spectra. If $\mathbb{C}$ is split exact, the THH and TC spectra are $\Omega$-spectra. The inclusion by degeneracies $T H H_{0}\left(S^{(n)} \mathbb{C}\right) \subseteq T H H\left(S^{(n)} \mathbb{C}\right)$ is a stable equivalence, and it is shown how this leads to a weak resolution theorem for THH. If $\mathscr{P}_{A}$ is the category of finitely generated projective modules over a unital and associative ring $A$, we get that $T H H(A) \stackrel{\approx}{\rightarrow} T H H\left(\mathscr{P}_{A}\right)$ and $T C(A) \underset{ }{\rightrightarrows} T C\left(\mathscr{P}_{A}\right)$.


The Dennis trace map from algebraic K-theory of a ring generated an interest for Hochschild homology from a K-theoretic point of view. Bökstedt defined in [2] a factorization of the Dennis trace through a theory reminiscent of Hochschild homology, but with a much richer structure. He also made calculations on examples which gave new K-theoretic information, and called the new theory topological Hochschild homology (THH). The factorization had earlier been suggested by Goodwillie, and it is known to coincide with the linearization of K-theory in the sense of calculus of functors (see [7], and also the program of R. Schwänzl, R. Staffeldt and F. Waldhausen).

As defined by Bökstedt, topological Hochschild homology comes equiped with a cyclic action. This cyclic action plays an important role, e.g. in the work of Bökstedt et al. [3] on the K-theory analogue of the Novikov conjecture. In fact, the map from

[^0]K-theory factors through the fixed points of the various operations on THH arising from this cyclic structure, and comparison with K-theory should take this into account.

Algebraic K-theory is defined in terms of exact categories (or more generally by categories with cofibrations and weak equivalences), and in this paper we adopt the point of view that $T H H$ should be defined on this level. This has both advantages and disadvantages. The most obvious disadvantage is that the theory becomes more involved. One of the advantages is that the additivity of the category can be used constructively on the theory itself. Another advantage is that the map from K-theory is very transparent. Pirashivili and Waldhausen [12] give a model for $T H H$ of rings which is defined in terms of the category of finitely generated projective modules over the ring. A variant of this model proved itself useful in [7] where both the advantages mentioned above were crucial tools. This only disadvantage of these constructions is that they admit no interesting cyclic action.

The present paper aims at giving a model for $T H H$ which combines the categorical outlook with the presence of the cyclic actions.

This paper is somewhat encyclopedic, but is meant to contain the results needed for the computations on exact categories in Chapter 2 (all but one of the general calculations are needed at some point in this example), as well as a series of intended future applications. Most notably, McCarthy has already proven that the result of Goodwillie of nilpotent extensions is true when reinterpreted properly. The setting should also be general enough to allow for applications. directed towards algebraic K-theory of spaces, and so the "linear" viewpoint of [7] cannot be adopted.

Thus the first goal should be: Define a topological Hochschild homology which meets the following requirements,
(1) It should agree with the definition of Bökstedt if given the same input.
(2) It should be defined in terms of categories and generally be equipped with an appropriate cyclic structure. Rationally it should agree with a Hochschild-type homology.
(3) It should make sense to "mix" K-theory and topological Hochschild homology, and the iteration of the $S$ construction should yield a spectrum corresponding to the K-theory spectrum.
(4) It should respect "Morita equivalence". In particular, if $A$ is a ring and $\mathscr{P}_{A}$ is the category of finitely generated projective $A$ modules, then the theory applied to $\mathscr{P}_{A}$ should correspond to the theory applied to $A$ itself.
(5) There should be a simple map from algebraic K-theory mapping into the fixed points of the cyclic actions, and agreeing with the cyclotomic trace. The map should be transparent enough to allow simple comparisons.
Most of the paper is devoted to showing that we indeed have a well defined theory satisfying these demands.

In order to do so, we introduce the concept of a ring functor on a category $\mathscr{C}$ which is roughly an FSP (Functor with Smash Product) with several objects (namely the ones in $\mathscr{C}$ ). When this is done, the construction of THH meeting (1) and (2) is straightforward (up to a choice or two).

To demonstrate that THH obeys the remaining demands we must prove a number of standard theorems. For instance, THH must commute with products, behave well under formation of matrices ("Morita equivalence") and operations similar to the $S$ construction ("triangular matrices"). This is proved directly by means of displaying actual homotopies between suitably simplified models.

Just as for FSPs, our model possesses a "Frobenius action" (see 1.5), and we define the topological cyclic homology $T C$ (as in [3]), as the inverse homotopy limit under the inclusion of fixed points and the Frobenius maps.

The example treated in Chapter 2 is that of exact categories. More precisely: suppose $\mathbb{C}$ is an exact category. Then we may consider the ring functor on $\mathbb{C}$ which sends a pair $(a, b) \in \mathbb{C}^{0} \times \mathbb{C}$ and a pointed simplicial set $X$ to $\mathbb{C}(a, b) \otimes Z[X] / Z[*]$ where $Z[X]$ is the free simplicial abelian group on $X$. This makes $T H H$ and $T C$ into functors from the category of exact categories. More generally, if $S \mathbb{C}$ is the $S$ construction of Waldhausen [17], we may apply $T H H$ or $T C$ to $S \mathbb{C}$. These theories are theoretically better behaved than simply applying the functor directly to the categories. In this case various simplifications are possible. For one thing we show directly that the models used in [7] agree with the present definitions. We note that, in accordance with the general idea, some typical K -theoretic properties such as confinality are obeyed by $T H H$. Another point worth mentioning is that the model for THH of an exact category is actually dependent upon the choice of exact sequences, but that if we apply THH to the underlying additive category (with just s plit exact sequences) then this theory agrees with the definition of the homology of a category as in [1]. The proof that $T H H(A) \simeq T H H\left(\mathscr{P}_{A}\right)$ could have been varied by the use of a construction which associates an FSP to any ring functor in a manner keeping $T H H$ unchanged.

Summarizing, the main results of Chapter 2 are
(1) There is a simple map from K-theory of an exact category $\mathbb{C}$ into $\Omega T H H(S \mathbb{C})$, mapping into the fixed points of both the cyclic actions and the Frobenius maps, and hence there is a lifting to $\Omega T C(S \mathbb{C})$. This forms a map of spectra upon iterating the $S$ construction.
(2) For split exact categories $T H H(\mathbb{C})$ and $\Omega T H H(S \mathbb{C})$ are equivalent (likewise for TC).
(3) For additive categories the homology of the category $\mathbb{C}$ itself, $H_{\boldsymbol{k}}(\mathbb{C}, \mathbb{C})$, is isomorphic to $\pi_{k} T H H(\mathbb{C})$.
(4) The model for $T H H$ in [7] is equivalent to the present via a simple map.
(5) For a ring $A$ we have an equivalence $\operatorname{THH}(A) \simeq T H H\left(\mathscr{P}_{A}\right)$ (likewise for TC).
(6) A weak version of the resolution theorem is valid for THH.

# 1. Topological Hochschild homology of ring functors on a category 

### 1.0. Introduction

Analogously to Bökstedt's definition of a functor with smash product (FSP) we introduce the notion of a ring functor on a category. Roughly the difference is the same as the difference between a ring with unit and a linear category. That is, a ring functor on a category with only one object is an FSP. We will also need the notion of a module over a ring functor. Important examples will be given in the following section. One may also define the topological Hochschild homology (THH) for ring functors and the remainder of the chapter is dedicated to establishing the basic properties of $T H H$. It turns out that most of what you can do for FSPs you can do for ring functors. The main interest in this generalization is that the comparison with K-theory becomes simpler. For instance in the case of rings, the point is that instead of working with the group completion of spaces of matrices mapping by devious routes into the topological Hochschild homology of the ring, we may now consider a simple map into the topological Hochschild homology of the projective modules, or even better into a mixing of $T H H$ and algebraic K-theory. The goal of this chapter is to prove that the range of such maps exists and is well behaved. We will come back to the question whether the theory is comparable (hint: it is!) with the classical case in the text chapter.

The model we present comes with all the usual cyclic structure, and will take pains to show when this structure is preserved. We do this in a somewhat unusual way by restricting our attention to simpler models for $T H H$ not containing degeneracies. This requires some care, but the gain in having simple model outweighs the disadvantages as there is essentially only a single trick (more precisely: Lemma 1.5.12) you have to apply repeatedly.

The main results in this chapter are:
(1) For any ring functor $A$ and $A$-bimodule $P$ the $T H H(A, P)$ is a well defined object, with cyclic structure if $P=A$.
(2) Rationally $T H H(A, P)$ is equivalent to an additive cyclic nerve.
(3) THH is well behaved under natural isomorphisms and equivalences, respects direct limits, stable equivalences and preserves products.
(4) Morita equivalence is true for $T H H$.
(5) THH may be calculated by means of THH of an FSP, but not functorially. As to the last point it should be noted that the FSP in question rarely will be of a sort which invites closer analysis.
1.0.1. Some language. By a linear category we mean a category whose morphism sets are abelian groups and whose composition is bilinear, or what is often called a pre-additive category. Given any category $\mathscr{C}$ we may from the free linear category on $\mathscr{C}$, which we will call $\boldsymbol{Z} \mathscr{C}$, with the same objects as $\mathscr{C}$, but with morphism sets
$\boldsymbol{Z} \mathscr{C}(a, b)=\boldsymbol{Z}[\mathscr{C}(a, b)]:$ the free abelian group on the set $\mathscr{C}(a, b)$ of morphisms from $a$ to $b$ in $\mathscr{C}$. An additive category is a linear category with a zero object and finite products.

Given two categories $\mathscr{C}$ and $\mathscr{D}$ we will let $\mathscr{D}^{\mathscr{C}}$ denote the category of functors from $\mathscr{C}$ to $\mathscr{D}$ and natural transformations. Let $\mathscr{C}$ be a category. We will write $\mathscr{C}^{\circ}$ for the opposite category; i.e. the category with the same objects as $\mathscr{C}$ but with all arrows reversed. We will write $\mathscr{C}(-,-)$ for the functor from $\mathscr{C}^{0} \times \mathscr{C}$ assigning to each pair of objects $a$ and $b$ in $\mathscr{C}$ the set of morphisms from $a$ to $b$. For convenience in these notes we will often write $T_{0} \mathscr{C}$ for $\mathscr{C}^{\circ} \times \mathscr{C}$. Let $T_{1} \mathscr{C}$ be the full subcategory of $T_{0} \mathscr{C} \times T_{0} \mathscr{C}$ with objects of the form $((a, b),(c, a))$ and let composition be the functor $T_{1} \mathscr{C} \rightarrow T_{0} \mathscr{C}$ induced by $((a, b),(c, a)) \mapsto(a, b) \circ(c, a)=(c, b)$. In view of the convention we will often write objects in $T_{0} \mathscr{C}=\mathscr{C}^{\circ} \times \mathscr{C}$ by single letters like $f$ and think of them as arrows. More generally we let $T_{k} \mathscr{C}$ be the full subcategory of $\left(\mathscr{C}^{\circ} \times \mathscr{C}\right)^{k+1}$ with objects on the form $\left(\left(c_{1}, c_{0}\right),\left(c_{2}, c_{1}\right), \ldots,\left(c_{k+1}, c_{k}\right)\right)$ (called composable $k+1$-tuples). This notation is inspired by the twisted arrow category $T \mathscr{C}$ whose objects are the morphisms of $\mathscr{C}$ and where an arrow from $f: a \rightarrow b$ to $g: c \rightarrow d$ is a commutative diagram


The forgetful functor $T \mathscr{C} \rightarrow T_{0} \mathscr{C}$ sends a morphism $f: a \rightarrow b$ to $(a, b)$. The definitions of ring functors and modules are based on $T_{0} \mathscr{E}$, but could equally well (perhaps better) have used $T \mathscr{C}$; however at current all important examples are covered by the present definition.
1.0.2. Simplicial objects. Let $\Delta$ be a category of standard ordered finite sets $[n]=\{0<1<\cdots<n\}$ and monotone maps. A simplicial object in a category $\mathscr{C}$ is a functor from $\Delta^{\circ}$ to $\mathscr{C}$. The category of simplicial objects in $\mathscr{C}$ will be denoted $s \mathscr{C}$. We will let $s_{*} \mathscr{E} n s$ (resp. $f s_{*} \mathscr{E} n s$ ) denote the category of pointed simplicial sets (resp. finite pointed simplicial sets). $\Delta(n)$ denotes the pointed simplicial set $\{[q] \mapsto \Delta([q],[n])\}$ pointed at the zero map. As models for the spheres we will choose $S^{n}=\bigwedge_{n \text {-fold }} S^{1}$ where $S^{1}=\Delta(1) / \partial \Delta(1)$. Unless otherwise stated, by " $\Omega^{n} X^{\prime}$ " we will mean sing $|X|^{\left|S^{n}\right|}$ where $\mid$ | is the realization and "sing" is the singular complex of a topological space. Given any simplicial set $X$ we will write $Z[X]$ for the simplicial abelian group which in degree $q$ is $Z\left[X_{q}\right]$, the free abelian group on $X_{q}$. If $X$ is pointed, $\tilde{Z}[X]$ denotes the quotient $Z[X] / Z[*]$. We will say that a pointed simplicial set $X$ is $n$-connected if $\pi_{q}(X)=0$ for all $q \leq n$, and if $n \geq 0$ that a map $f: X \rightarrow Y$ is $n$-connected if the homotopy fiber is $n-1$ connected and $\pi_{0}(X) \rightarrow \pi_{0}(Y)$ is surjective. A pointed simplicial set $X$ is $n$-reduced if $X_{q}=*$ for all $q \leq n$.
1.0.3. Simplicial categories. A simplicial category is an object of the functor category from $\Delta^{\circ}$ to some category of categories. A simplicial functor is a natural transformation between two simplicial categories. If a pointed simplicial category $\mathscr{C}$ has
a representation of the functor $s_{*} \mathscr{E} n s(X, \mathscr{C}(c,-))$, say a natural equivalence $\mathscr{C}(X \otimes c,-) \cong s_{*} \mathscr{E} n s(X, \mathscr{C}(c,-))$ where $c \in \mathscr{C}$ and $X \in s_{*} \mathscr{E} n s$, we will say that $\mathscr{C}$ has products with pointed simplicial sets. Likewise we say that $\mathscr{C}$ has products with finite pointed simplicial sets if $s_{\boldsymbol{*}} \mathscr{E} n s(X, \mathscr{C}(c,-))$ has a representation for every $X \in f s_{\boldsymbol{*}} \mathscr{E} n s$.

Suppose that the pointed simplicial categories $\mathscr{C}$ and $\mathscr{D}$ have products with (finite) pointed simplicial sets. Then any pointed simplicial functor $F^{\prime}: \mathscr{C} \rightarrow \mathscr{D}$ has associated to it a natural transformation $\lambda_{x, c}: X \otimes F(c) \rightarrow F(X \otimes c)$ induced by the identity via


In particular case of the pointed simplicial category $s_{*} \mathscr{E} n S^{\mathscr{*}}$ where $\mathscr{B}$ is any category we have a product defined for any $G \in s_{*} \mathscr{E} n s^{\mathscr{I}}$ and $X \in s_{*} \mathscr{E} n s$ by sending $b \in \mathscr{B}$ to $(X \wedge G)(b)=X \wedge G(b)$.

### 1.1. Ring functors on a category

In analogy with the definitions of an FSP (see [2], [3], [8] or [12]) we now define ring functors on a category.
1.1.1. Definition. For $\mathscr{C}$ a small category, let $L \mathscr{C}$ be the full subcategory of the category of pointed simplicial functors $F: f s_{*} \mathscr{E} n s \rightarrow s_{*} \mathscr{E} n s^{T_{0} \mathscr{C}^{\mathscr{C}}}$ such that if $X$ is $n$ connected and $f=(a, b) \in T_{0} \mathscr{C}=\mathscr{C}^{\circ} \times \mathscr{C}$
(1) $F(X)(f)$ is $n$-connected,
(2) the map $S^{1} \wedge F(X)(f) \rightarrow F\left(S^{1} \wedge X\right)(f)$ induced from the simplicial structure is $2 n-c$ connected for some number $c$ not depending on $X$.

We will denote the natural transformation due to the simplicial structure discussed above by

$$
\lambda_{X, Y}: X \wedge F(Y) \rightarrow F(X \wedge Y)
$$

where $X \wedge F(Y)$ sends $f \in T_{0} \mathscr{C}$ to $X \wedge F(Y)(f)$.
Remark. One may weaken condition (1) to require that $F(X)(a, b)$ is $n-d^{F}(a)+c^{F}(b)$ connected, where $d^{F}(a)$ and $c^{F}(b)$ are non-negative numbers depending only on $a$ and $b . d^{F}$ may be thought of as dimension and $c^{F}$ as connectivity. This weakening is useful e.g. if $\mathscr{C}$ is some category of spaces.

A spectrum (often called pre-spectrum) is a sequence of spaces $X^{m}$ with maps $S^{1} \wedge X^{m} \xrightarrow{f^{m}} X^{m+1}$, and an $\Omega$-spectrum is a spectrum where the adjoints of the
structure maps yield homotopy equivalences $X^{m} \simeq \Omega X^{m+1}$. A map of spectra is a stable weak equivalence if it induces an isomorphism on all homotopy groups (as spectra). We may consider $F$ as a functor from $T_{0} \mathscr{C}$ to spectra via $\underline{\underline{F}}(f)=\left\{m \mapsto F^{m}(f)=F\left(S^{m}\right)(f)\right\}$ with structure maps $\lambda_{S^{1} . s^{m}}$.

A morphism $\phi: F \rightarrow G$ in $L \mathscr{C}$ is called a stable equivalence if it induces a stable weak equivalence $\underline{\underline{F}}(f) \rightarrow \underline{\underline{G}}(f)$ for each $f \in T_{0} \mathscr{C}$.
1.1.2. Definition (Ring functors). Let $\mathscr{C}$ be a small category. A ring functor on $\mathscr{C}$ is an object $A \in L \mathscr{C}$ together with a natural transformation which we will call the multiplication

$$
\mu_{X, Y}(f, g): A(X)(f) \wedge A(Y)(g) \rightarrow A(X \wedge Y)(f \circ g)
$$

for every composable pair $(f, g) \in T_{1} \mathscr{C}$, such that multiplication $\mu$ is strictly associative. More precisely

commutes for all $X, Y, Z \in f s_{*} \mathscr{E} n s$ and composable triples $(f, g, h)$.

To be entirely clear: $\mu$ is a natural transformation from the composite

$$
f s_{*} \mathscr{E} n s \times f s_{*} \mathscr{E} n s \xrightarrow{A \times A} s_{*} \mathscr{E} n S^{T_{0} ष} \times s_{*} \mathscr{E} n s^{T_{0} \&} \xrightarrow{\wedge \exists} s_{*} \mathscr{E} n s^{T_{1} \mathscr{E}}
$$

to

$$
f s_{*} \mathscr{E} n s \times f s_{*} \mathscr{E} n s \xrightarrow{\wedge} s_{*} \mathscr{E} n s \xrightarrow{A} s_{*} \mathscr{E} n s^{T_{0} \mathbb{E}} \rightarrow s_{*} \mathscr{E} n s^{T_{1} \mathscr{E}}
$$

where the latter functor is induced by the composition $T_{1} \mathscr{C} \rightarrow T_{0} \mathscr{C}$.
There is a particularly important ring functor

$$
S: f s_{*} \mathscr{E} n s \rightarrow s_{*} \mathscr{E} n s^{T_{0} ष}
$$

namely the functor sending $X \in s_{*} \mathscr{E} n s$ to the constant functor $f \in T_{0} \mathscr{C} \mapsto X$. The multiplication is simply the identity. We will call this ring functor the identity ring functor $S$. One may think of this ring functor as the analogue of the ring of integers.

If $\mathscr{C}$ is a category, we consider the associated discrete subcategory $d \mathscr{C} \subseteq \mathscr{C}$ consisting of all objects in $\mathscr{C}$, but only the identity morphisms. There is a functor $D: d \mathscr{C} \rightarrow T_{0} \mathscr{C}$ given by the diagonal. Hence there is a functor $s_{*} \mathscr{E} n s^{T_{0} \mathscr{8}} \xrightarrow{D^{*}} s_{*} \mathscr{E} n s^{d \mathscr{C}}$.
1.1.3. Definition (Unital ring functors). Let $A$ be a ring functor on $\mathscr{E}$, and let as above $S$ be the identity ring functor on $\mathscr{C}$. We say that $A$ is a ring functor with unit on $\mathscr{C}$, or
simply that $A$ is unital, if there is a natural transformation 1 from $D^{*} \circ S$ to $D^{*} \circ A$ such that for every $X, Y \in f s_{*} \mathscr{E} n s$ and $f=(a, c) \in T_{0} \mathscr{C}$

commute, and likewise for $\mu \circ(i d \wedge 1)$ up to a switch of factors.

The reader may wonder why one has to restrict to the discrete subcategory in order to define the unit, but the reason is natural enough: only when range and target coincides should there be a distinguished "identity morphism".

In particular the identity ring functor $S$ on $\mathscr{C}$ is unital.
1.1.4. Change of underlying category. Given a small category $\mathscr{C}$ we let $\mathscr{F} \mathscr{C}$ (resp. $\mathscr{F} \mathscr{C}^{u}$ ) be the category with objects ring functors (resp. unital ring functors) on $\mathscr{C}$ and morphisms transformations in $L \mathscr{C}$ compatible with the multiplicative structure (and unit).

If $\phi: \mathscr{C} \rightarrow \mathscr{D}$ is a functor we define $\phi^{*}: \mathscr{F} \mathscr{D} \rightarrow \mathscr{F} \mathscr{C}$ by composition. That is, if $A$ is a ring functor on $\mathscr{D}$ we let $\phi^{*} A$ be the composite of $A: f s_{*} \mathscr{E} n s \rightarrow s_{*} \mathscr{E} n s^{\mathscr{D}^{\circ} \times \mathscr{D}}$ and $\phi^{*}: s_{*} \mathscr{E} n s^{\mathscr{D}^{\circ} \times \mathscr{D}} \rightarrow s_{*} \mathscr{E} n s^{\mathscr{C}^{\circ} \times \mathscr{8}}$. This clearly is also well defined in the unital case.

Isomorphic functors $\phi \cong \psi: \mathscr{C} \rightarrow \mathscr{D}$ induced isomorphic functors $\phi^{*} \cong \psi^{*}$. It is worthwhile to spell this out explicitly: assume $\eta: \phi \rightarrow \psi$ is an isomorphism. Then

$$
A(X)\left(\eta(a)^{-1}, \eta(b)\right): A(X)(\phi(a), \phi(b)) \rightarrow A(X)(\psi(a), \psi(b))
$$

defines the desired isomorphism between $\phi^{*} A$ and $\psi^{*} A$. As this isomorphism is defined by isomorphisms in $\mathscr{D}$ it carries over to morphisms between ring functors and we get that $\phi^{*}$ and $\psi^{*}$ are isomorphic as functors.

Perhaps a note as to how to define ring functors over the twisted arrow category is appropriate here. The definition of ring functors then appears exactly as above, exchanging $T_{0} \mathscr{C}=\mathscr{C}^{0} \times \mathscr{C}$ with $T \mathscr{C}$ everywhere (the exposition was originally intended this way). As to the unital case, use the functor $d \mathscr{C} \rightarrow T \mathscr{C}$ sending an object to its identity map, and an identity map to the corresponding square with all sides the identity. Over $T \mathscr{C}$, a unital ring functor will then require a natural transformation from the identity functor under restriction to $d \mathscr{C}$.

### 1.1.5. Modules of ring functors

1.1.6. Definitions (Modules). Let $\mathscr{C}$ be a category, $A$ be a ring functor on $\mathscr{C}$ and $P \in L \mathscr{C}$. A left module structure on $P$ is a natural transformation

$$
\ell_{X, Y}(f, g): A(X)(f) \wedge P(Y)(g) \rightarrow P(X \wedge Y)(f \circ g)
$$

for any composable pair $(f, g) \in T_{1} \mathscr{E}$, such that

commutes. If $A$ is unital, we define a unital left module structure on $P$ as a left module structure on $P$ such that

commutes. A structure of a (unital) right module is defined similarly.
A bimodule structure on $P$ (resp. unital bimodule structure in the case $A$ is unital) consists of compatible left and right module structures on $P$ (resp. compatible unital left and right module structures); i.e., for all composable triples ( $f, g, h$ ) the following diagram commutes:


We will refer to such $P$ together with this structure as an $A$ bimodule.
Note that all $T \in L \mathscr{C}$ are automatically unital $S$ bimodules due to the simplicial structure.

In the case of the extension where the connectivity assumption (1) in the definition of $L \mathscr{C}$ was weakened we will in addition require of an $A$ module $P$ that for all $a \in \mathscr{C}$ we have that $d^{P}(a) \leq d^{A}(a)$. The extension to the twisted arrow category is straightforward. If $\phi: \mathscr{D} \rightarrow \mathscr{C}$ is any functor we define the $\phi^{*} A$ module $\phi^{*} P$ in the same manner as $\phi^{*} A$ with the obvious actions.
1.1.7. Notation. If $A$ is a ring functor on some category $\mathscr{C}$ we will often write $A^{X}(c, d)$ instead of $A(X)(c, d)$ when the emphasis is rather on the category than on the simplicial set. In particular, we write $A^{n}(c, d)$ for $A\left(S^{n}\right)(c, d)$. Similarly, if $P$ is an $A$ module we write $P^{X}(c, d)$ for $P(X)(c, d)$ and $P^{n}(c, d)$ for $P\left(S^{n}\right)(c, d)$.

### 1.2. Examples

We now list some useful ring functors. Sections 1.2.1-1.2.6 contain the most important ones. After that things develop into more of a bestiary, listed here only to have a convenient place to refer back to as the various constructions are needed later on. Many of the examples have analogues for bimodules, but as the list is already long, we leave that to the interested reader.

From the examples below one might be tempted to call ring functors on a category "FSPs with several objects" in analogy with "linear categories are rings with several objects" [11]. Note, however, that a ring functor on $\mathscr{C}$ is not determined by its underlying category. For instance, a simplicial category gives rise to (at least) two ring functors on $\mathscr{C}$ via examples 1.2.2 and 1.2.3.
1.2.1. FSPs. Any FSP $A$ gives rise to a "constant" ring functor on a given category $\mathscr{C}$ via $A(X)(a, b)=A(X)$. If $\mathscr{C}$ is the trivial category, the notions of FSPs and unital ring functors on $\mathscr{C}$ naturally coincide. Note that if $A$ is a unital ring functor on a one point category it is simply an FSP with extra structure. In fact, the natural examples of FSPs come with such an extra structure (see Examples 1.2.2 and 1.2.3 restricted to one point categories). In these notes we call ring functors on one point categories simply FSPs whether they have units or not.
1.2.2. The half smash ring functor on a simplicial category. Let $\mathscr{C}$ be a simplicial category. Then $X \mapsto \mathscr{C}(-,-)_{+} \wedge X$ is a unital ring functor. The multiplication is induced by composition via

$$
\begin{aligned}
& \left(\mathscr{C}(a, b)_{+} \wedge X\right) \wedge\left(\mathscr{C}(c, a)_{+} \wedge Y\right) \\
& \quad \cong(\mathscr{C}(a, b) \times \mathscr{C}(c, a))_{+} \wedge(X \wedge Y) \rightarrow \mathscr{C}(c, b)_{+} \wedge(X \wedge Y)
\end{aligned}
$$

and the unit by the inclusion $X \xrightarrow{x_{\mapsto} i d_{a} \wedge x} \mathscr{C}(a, a)_{+} \wedge X$. Later we will show that the homotopy type of the topological Hochschild homology of this ring functor is simply the stabilization of the cyclic nerve of $\mathscr{C}$. When $M$ is a simplicial monoid, regarded as a category with only one object in each dimension, we recover the FSP of [2] associated to the monoid.
1.2.3. The linear ring functor on a linear category. Let $\mathscr{C}$ be a (simplicial) linear category. Then

$$
X \mapsto X \mapsto \mathscr{C}(-,-) \otimes_{Z} \tilde{Z}[X]
$$

is a unital ring functor on $\mathscr{C}$ where $\tilde{Z}[X]=\boldsymbol{Z}[X] / \boldsymbol{Z}[*]$. The multiplication is given by sending smash to tensor followed by composition:

$$
\begin{aligned}
& \left(\mathscr{C}(a, b) \otimes_{Z} \tilde{Z}[X]\right) \wedge\left(\mathscr{C}(c, a) \otimes_{Z} \tilde{Z}[Y]\right) \\
& \quad \rightarrow(\mathscr{C}(a, b) \otimes \mathscr{C}(c, a)) \otimes_{Z} \tilde{Z}[X \wedge Y] \rightarrow \mathscr{C}(c, b) \otimes_{Z} \tilde{Z}[X \wedge Y]
\end{aligned}
$$

and the unit by the inclusion $X \xrightarrow{x \mapsto i d_{d} \otimes 1 \cdot x} \mathscr{C}(a, a) \otimes_{Z} \tilde{Z}[X]$.
The above construction extends to all categories $\mathscr{C}$ by passing to the free linear category $\boldsymbol{Z} \mathscr{C}$, or more directly consider the ring functor on $\mathscr{C}$ (which also is a ring functor on $\boldsymbol{Z} \mathscr{E}$ )

$$
X \mapsto Z \mathscr{C}(-,-) \otimes \tilde{Z}[X]
$$

Note that this may also be obtained from 1.2 .2 as $Z \mathscr{C}(a, b) \otimes \tilde{Z}[X] \cong$ $\tilde{Z}\left[\mathscr{C}(a, b)_{+} \wedge X\right]$. If $R$ is a ring with unit considered as a linear category with only one object then we recover the definition of the topological Hochschild theory of a ring given in [2].
1.2.4. FSPs associated to general ring functors. The incidence FSP. Given a ring functor $A$ on a category $\mathscr{C}$ it determines an FSP [A] given by

$$
[A](X)=\prod_{a \in \mathscr{C}} \bigvee_{b \in \mathscr{C}} A^{X}(a, b)
$$

The multiplication is given by matrix multiplication:

$$
\begin{gathered}
{[A](X) \wedge[A](Y)=\left(\prod_{a \in c} \vee_{b \in c} A^{X}(a, b)\right) \wedge\left(\prod_{c \in c} \vee_{d \in c} A^{Y}(c, d)\right)} \\
\Pi_{c \in c} \vee_{d \epsilon c}\left(\left(\Pi_{a \in c} \vee_{b \in c} A^{X}(a, b)\right) \wedge A^{Y}(c, d)\right) \\
\Pi \vee\left(p r_{d} \wedge d\right) \mid \\
\Pi_{c \epsilon c} \vee_{d \in c}\left(\vee_{b \in c} A^{X}(d, b) \wedge A^{Y}(c, d)\right) \\
\Pi \vee \vee \mu \mid \\
\Pi_{c \in c} \vee_{d \in c}\left(\vee_{b \in c} A^{X \wedge Y}(c, b)\right) \\
\text { fold over } d \mid \\
\Pi_{c \in c} \vee_{b \in c} A^{X \wedge Y}(c, b)=[A](X \wedge Y)
\end{gathered}
$$

If $A$ has unit $[A]$ is unital and the unit is given by the diagonal

$$
X \xrightarrow{\text { the diagonal }} \prod_{c \in \mathscr{\mathscr { E }}} X \xrightarrow{\Pi 1_{x}(c)} \prod_{c \in \mathscr{\mathscr { C }}} A^{X}(c, c) \subseteq[A](X) .
$$

There is another FSP associated to $A$ of particular interest, namely $[A]_{\vee}$ given by

$$
[A]_{\vee}(X)=\bigvee_{(a, b) \in \mathscr{Y}^{2}} A^{X}(a, b)
$$

and multiplication similar to (but slightly easier than) [A]. The drawback of this FSP is that it has no unit, but it still is much easier to work with than [ $A$ ] as it is equipped with a "sum" (fold). Whenever $\mathscr{C}$ is equivalent to the direct limit of its finite subcategories the inclusion $[A]_{\vee} \rightarrow[A]$ is a stable equivalence by the Blakers-Massey triad connectivity theorem.

In the special case where $A$ is a ring functor with values in simplicial abelian groups with bilinear multiplication it determines an FSP $[A]_{\oplus}$ given by

$$
[A]_{\oplus}(X)=\bigoplus_{(a, b) \in \mathscr{C}^{2}} A^{X}(a, b) .
$$

The multiplication is given by

$$
\begin{aligned}
& {[A]_{\oplus}(X) \wedge[A]_{\oplus}(Y)=\bigoplus_{(a, b) \in \mathscr{C}^{2}} A^{X}(a, b) \wedge \bigoplus_{(c, d) \in \mathscr{Y}^{2}} A^{Y}(c, d)} \\
& \longrightarrow \bigoplus_{(a, b) \in \mathscr{Q}^{2}} A^{X}(a, b) \otimes \underset{(c, d) \in \mathscr{E}^{2}}{ } A^{Y}(c, d) \\
& \longrightarrow \bigoplus_{(a, b, c) \in \mathbb{母}^{3}} A^{X}(a, b) \otimes A^{Y}(c, a) \\
& \xrightarrow{\mu} \underset{(a, b, c) \in \mathscr{C}^{3}}{\oplus} A X \wedge Y(c, b) \xrightarrow{\text { sum }} \bigoplus_{(c, b) \in \mathscr{Q}^{2}} A^{X \wedge Y}(c, b)
\end{aligned}
$$

where the second arrow sends $A^{X}(a, b) \otimes A^{Y}(c, d)$ to the basepoint if $d \neq a$. An important difference between $[A]$ and $[A]_{\oplus}$ is that if $A$ is unital then $[A]_{\oplus}$ is unital only if $\mathscr{C}$ is finite (i.e. has only finitely many objects). The unit is still given by the diagonal

$$
X \xrightarrow{\text { the diagonal }} x_{c \in \mathscr{E}} X \xrightarrow{\times 1_{x}(c)} x_{t \in \mathscr{C}} A^{X}(c, c) \subseteq[A]_{\oplus}(X) .
$$

There is a variant of $[A]_{\oplus}$ where we allow all row finite matrices. This is more analogous to [A] above and in this case there is a unital stable equivalence from [A]. On the other hand the map $[A]_{\vee} \rightarrow[A]_{\oplus}$ is also a stable equivalence, but this has the disadvantage of not being unital even for finite categories (with more than one object).
1.2.5. The product and sum of two ring functors. Let $\mathscr{C}_{1}$ and $\mathscr{C}_{2}$ be small categories. Given $A_{1} \in \mathscr{F} \mathscr{C}_{1}$ and $A_{2} \in \mathscr{F} \mathscr{C}_{2}$ we may from their product $A_{1} \times A_{2} \in \mathscr{F}\left(\mathscr{C}_{1} \times \mathscr{C}_{2}\right)$ by

$$
\left(A_{1} \times A_{2}\right)^{X}\left(\left(c^{1}, c^{2}\right),\left(d^{1}, d^{2}\right)\right)=A_{1}^{X}\left(c^{1}, d^{1}\right) \times A_{2}^{X}\left(c^{2}, d^{2}\right)
$$

with componentwise multiplication. If both $A_{1}$ and $A_{2}$ are unital so is $A_{1} \times A_{2}$ and the unit map is induced by the diagonal

$$
X \rightarrow X \times X \xrightarrow{\mathbf{1}_{x}\left(c^{1}\right) \times 1_{x}\left(c^{2}\right)} A_{1}^{X}\left(c^{1}, c^{1}\right) \times A_{2}^{X}\left(c^{2}, c^{2}\right)
$$

When $\mathscr{C}=\mathscr{C}_{1}=\mathscr{C}_{2}$ we may of course compose this with the diagonal diag: $\mathscr{C} \rightarrow \mathscr{C} \times \mathscr{C}$ to obtain the new ring functor on $\mathscr{C}, \operatorname{diag} *\left(A_{1} \times A_{2}\right)$, which we will call the internal product.

Similarly we may define a sum of $A_{1} \in \mathscr{F} \mathscr{C}_{1}$ and $A_{2} \in \mathscr{F} \mathscr{C}_{2}$ by

$$
\left(A_{1} \vee A_{2}\right)^{X}\left(\left(c^{1}, c^{2}\right),\left(d^{1}, d^{2}\right)\right)=A_{1}^{X}\left(c^{1}, d^{1}\right) \vee A_{2}^{X}\left(c^{2}, d^{2}\right)
$$

with componentwise multiplication. Even when both $A_{1}$ and $A_{2}$ were unital, this generally has no unit. However, the map $A_{1} \vee A_{2} \rightarrow A_{1} \times A_{2}$ induced by the inclusion is compatible with the multiplicative structure. The considerations concerning the diagonal when $\mathscr{C}=\mathscr{C}_{1}=\mathscr{C}_{2}$ again applies here to give us the sum within ring functors on $\mathscr{C}$.
1.2.6. The $n \times n$ matrix of a ring functor. Let $\mathscr{C}$ be a small category, and let $n$ denote the set $\{1, \ldots, n\}$. Consider the $n$-fold product $\mathscr{C}^{n}$, and let $p r_{k}: \mathscr{C}^{n} \rightarrow \mathscr{C}$ be the $k$ th projection. We define the $n \times n$ matrix $M_{n} A$ of $A$ to be the ring functor on $\mathscr{C}^{n}$ given by

$$
\left(M_{n} A\right)^{X}(B, C)=\prod_{r \in n} \bigvee_{s \in n} A^{X}\left(p r_{r} B, p r_{s} C\right)
$$

for $X \in f s_{*} \mathscr{E} n s$ and $B, C \in \mathscr{C}{ }^{n}$. The multiplication is given by matrix multiplication, i.e.

where $B, C, D \in \mathscr{C}^{n}$ and $X, Y \in f s_{*} \mathscr{E} n s$. If $A$ has a unit so has $M_{n} A$ via

$$
X \xrightarrow{\text { the diagonal }} \prod_{r \in n} \xrightarrow{\Pi 1_{x}\left(p r_{r}(C)\right)} \prod_{r \in n} A^{X}\left(p r_{r}(C), p r_{r}(C)\right) \subseteq\left(M_{n} A\right)^{X}(C, C) .
$$

We also have the non-unital variant of the matrices using only sums, namely $\left(M_{n} A\right)_{V}$ given by

$$
\left(M_{n} A\right)_{V}^{X}(B, C)=\bigvee_{(r, s) \in n^{2}} A^{X^{\prime}}\left(p r_{r} B, p r_{s} C\right)
$$

and multiplication similar to (but easier than) $M_{n} A$. The inclusion $\left(M_{n} A\right)_{V} \rightarrow M_{n} A$ is a stable equivalence by Blakers Massey and condition (1) in Definition 1.1.

In the case where the ring functor has values in simplicial abelian groups with bilinear multiplication, we get matrices defined with $\oplus$ instead of $V$. Again this construction is unital if the original ring functor was. Furthermore there is a map compatible with the multiplicative structure from the matrices above to these additive matrices.
1.2.7. Upper triangular matrices. Let $A \in \mathscr{G} \mathscr{C} \mathscr{C}$. The upper triangular $n \times n$ matrix $T_{n} A$ is defined to be the subspace of the matrices given by

$$
\left(T_{n} A\right)^{X}(B, C)=\prod_{r \in n} \bigvee_{s \leq r \in n} A^{X}\left(p r_{r} B, p r_{s} C\right)
$$

with induced multiplication. This is again unital if $A$ is. There is also a stably equivalent version without unit using only wedges $\left(T_{n} A\right)_{V} \subseteq T_{n} A$ defined as for the matrices. In the linear case we may also define the upper triangular matrices with the sum. The same remarks apply to these upper triangular matrices as given for the full matrices.
1.2.8. Making the spectrum an $\Omega$ spectrum. Given a ring functor $A$ on a category $\mathscr{C}$ we may associate to it the ring functor on $\mathscr{C}$, denoted $A_{\Omega}$, given by

$$
A_{\Omega}^{X}(a, b)=\lim _{k \rightarrow \infty} \Omega^{k} A^{S^{\star} \wedge X}(a, b)
$$

and product induced by

$$
\begin{aligned}
& \Omega^{k} A^{S^{k} \wedge X}(a, b) \wedge \Omega^{l} A^{S^{\prime} \wedge Y}(c, a) \rightarrow \Omega^{k+l}\left(A^{S^{k} \wedge X}(a, b) \wedge A^{S^{l} \wedge Y}(c, a)\right) \\
& \quad \xrightarrow{\Omega^{k+i} \mu} \Omega^{k+l} A^{S^{k} \wedge S^{\prime} \wedge X \wedge Y}(c, b) .
\end{aligned}
$$

$A$ and $A_{\Omega}$ are stably equivalent ring functors. Furthermore, if $A$ has a unit so does $A_{\Omega}$ via

$$
X \rightarrow \lim _{k \rightarrow \infty} \Omega^{k}\left(S^{k} \wedge X\right) \xrightarrow{\lim \Omega^{k} 1_{s^{k}}(c)} A_{\Omega}^{X}(c, c) .
$$

1.2.9. The associated Eilenberg-MacLane ring functor. If $A$ is a ring functor on $\mathscr{C}$ we let $A_{E}$ be the ring functor on $\mathscr{C}$ given by

$$
A_{\mathrm{E}}^{X}(a, b)=\prod_{i=0}^{\infty} \pi_{i}(\underline{A}(a, b)) \otimes \tilde{Z}\left[S^{i} \wedge X\right]
$$

The multiplication is given by

where the first map simply maps smash of products to products of smashes, the second rearranges the terms and uses the product to induce one on the homotopy groups of the spectrum and the last one simply sums up. If $A$ is unital so is $A_{\mathrm{E}}$ via

$$
X \xrightarrow{\left[1_{s}(c)\right] \otimes \text { incl }} \pi_{0}(\underline{\underline{A}}(c, c)) \otimes \tilde{Z}[X] \rightarrow A_{\mathrm{E}}^{X}(c, c) .
$$

If the Postnikov invariants of the $A^{X}(a, b)$ are all zero we may choose a homotopy equivalence $A^{X}(a, b) \xrightarrow{\simeq} A_{E}^{X}(a, b)$.
1.2.10. The disjoint union. Let $\mathscr{C}_{1}$ and $\mathscr{C}_{2}$ be two categories and let $\mathscr{C}_{1} \amalg \mathscr{C}_{2}$ denote their disjoint union. If $A_{j}$ are ring functors on $\mathscr{C}_{j}, j=1,2$, we form the ring functor $A_{1} 山 A_{2}$ on $\mathscr{C}_{1} \amalg \mathscr{C}_{2}$ given by

$$
\left(A_{1} \amalg A_{2}\right)^{X}(a, b)= \begin{cases}A_{1}^{X}(a, b) & \text { if both } a, b \in \mathscr{C}_{1}, \\ A_{2}^{X}(a, b) & \text { if both } a, b \in \mathscr{C}_{2}, \\ * & \text { otherwise }\end{cases}
$$

The multiplication is given by

$$
\left(A_{1} \amalg A_{2}\right)^{X}(a, b) \wedge\left(A_{1} \amalg A_{2}\right)^{Y}(c, a) \rightarrow\left(A_{1} \amalg A_{2}\right)^{X}(c, a)
$$

mapping into the basepoint if not all $a, b, c$ are in a common component and otherwise using the appropriate multiplication. If both $A_{1}$ and $A_{2}$ are unital, so is $A_{1} 山 A_{2}$.

### 1.3. The topological Hochschild homology of a ring functor

In this section we will define the topological Hochschild homology of a ring functor on a category $\mathscr{C}$ in analogy with Bökstedt's definition [2] (see also [6]). As defined, this will be a presimplicial object. In the unitial case this will in fact be a simplicial
object, but it will for computational reasons be worthwhile to consider the general case. If we choose as our coefficient system (bimodule) the ring functor itself, this will be a (pre-) cyclic object. It will be of importance to us later to keep close track of these structures, so unfortunately this forces us to introduce some more language.
1.3.1. Presimplicial objects. As before, let $\Delta$ be the category of standard ordered finite sets $[n]=\{0<1<\cdots<n\}$ and monotone maps. Consider the subcategory $\Delta_{m} \subseteq \Delta$ with only injective maps. We call a functor from $\Delta_{m}^{0}$ to some category $\mathscr{C}$ a presimplicial object in $\mathscr{C}$. In [4] presimplicial sets are called $\Delta$-sets and in an earlier version of these notes they were called semi-simplicial. Natural transformations of such are called presimplicial maps. Note that all simplicial objects are by restriction to $\Delta_{m}$ presimplicial objects. Given a presimplicial object $X$ in a category with finite sums, we may form a simplicial object $\tilde{X}$ "by adjoining degeneracies" in the following way: let

Given $\psi \in \Delta([n],[q])$ and $\sigma$ in the $\phi$ summand of $\tilde{X}_{q}$, factor $\phi \psi=\eta \varepsilon$ canonically where $\varepsilon$ is a surjective and $\eta$ an injective map. Then $\psi^{*}(\sigma)=\eta^{*}(\sigma)$ in the $\varepsilon$ summand. In fact $X \mapsto \tilde{X}$ forms a left adjoint to the forgetful functor.
1.3.2. Precyclic objects. Let $\Lambda$ (resp. $\Lambda_{m}$ ) be the smallest subcategory of $\mathscr{E} n s$ containing $\Delta$ (resp. $\Delta_{m}$ ) and for each [ $\left.n\right]$ the extra morphism $\tau_{n}:[n] \rightarrow[n]$ given by cyclic permutation. A cyclic object in some category $\mathscr{C}$ is a functor from $\Lambda^{\circ}$ to $\mathscr{C}$, and a cyclic map is a natural transformation between two such functors. Similarly a precyclic object is a functor from $\Lambda_{m}^{\circ}$ and a precyclic map is a natural transformation between two precyclic objects. In an earlier version of these notes precyclic objects were called semi-cyclic.
1.3.3. The category I. Let $I$ be the category of standard finite sets $\{1, \ldots, n\}, n \geq 0$ (if $n=0$ this is the empty set) and injections. When there is no possibility of confusion we make no notational distinction between the sets and their cardinality. This category has more structure than the category of natural numbers $N$ considered as the subcategory with only standard inclusions. Most importantly $\left\{p \mapsto I^{p+1}\right\}$ forms a cyclic category with structure maps $\partial_{i}: I^{p+1} \rightarrow I^{p}, \sigma_{i}: I^{p} \rightarrow I^{p+1}$ and $\tau: I^{p} \rightarrow I^{p}$ given by

$$
\partial_{i}\left(x_{0}, \ldots, x_{p}\right)= \begin{cases}\left(x_{0}, \ldots, x_{i} \sqcup x_{i+1}, \ldots, x_{p}\right) & \text { for } 0 \leq i<p \\ \left(x_{p} \sqcup x_{0}, \ldots, x_{p-1}\right) & \text { for } i=p\end{cases}
$$

where $\sqcup$ means concatenation, $\sigma_{i}\left(x_{0}, \ldots, x_{p}\right)=\left(x_{0}, \ldots, x_{i}, 0, x_{i+1}, \ldots, x_{p}\right)$ and, $\tau\left(x_{0}, \ldots, x_{p}\right)$ $=\left(x_{p}, x_{0}, \ldots, x_{p-1}\right)$.
1.3.4. The topological Hochschild homology of a ring functor. We are now ready for the definition of our object of study. Bökstedt [3] defined the topological Hochschild homology of an FSP (see also [3], [8] and [12]). Analogous to the extension from

Hochschild homology of a ring to the additive cyclic nerve of a linear category we will now define the homology "THH" of a ring functor. One should note that in Examples 1.2.2 and 1.2.3 we recover Bökstedt's topological Hochschild homology of a simplicial monoid and a ring in case the categories in question have merely one object. More generally, if $A$ is any FSP considered as a ring functor over a one point category, then the present definition of $T H H(A)$ is the same as Bökstedt's.
1.3.5. Notation. If $\boldsymbol{x}=\left(x_{0}, \ldots, x_{p}\right) \in I^{p+1}$ we will set

$$
V(A, P)(x)=\bigvee_{\left(c_{0}, \ldots, c_{p}\right) \in \mathscr{C}^{p+1}} P^{x_{0}}\left(c_{0}, c_{p}\right) \wedge A^{x_{1}}\left(c_{1}, c_{0}\right) \wedge \cdots \wedge A^{x_{p}}\left(c_{p}, c_{p-1}\right)
$$

and write simply $V(A)(x)$ for $V(A, A)(x)$.
1.3.6. Definition (The topological Hochschild homology). Let $A$ be a ring functor on a small category $\mathscr{C}$ and let $P$ be a $A$ bimodule. We define $T H H(A, P): \Delta_{m}^{\circ} \rightarrow s_{*} \mathscr{E} n s$ to be the presimplicial object given by

$$
T H H_{p}(A, P)=\underset{x \in I^{p+1}}{\operatorname{holim}^{\sqcup x} V(A, P)(x)} \Omega^{\Delta x}
$$

where $\boldsymbol{x}=\left(x_{0}, \ldots, x_{p}\right)$ and $\bigsqcup_{\boldsymbol{x}}=x_{0} \sqcup x_{1} \sqcup \ldots \sqcup x_{p}$.
Face maps $d_{i}: \Omega^{\llcorner x} V(A, P)(x) \rightarrow \Omega^{\left\llcorner\partial_{i} x\right.} V(A, P)\left(\partial_{i} x\right)$ are defined as follows: $d_{0}$ is induced by

$$
\ell: P^{x_{0}}\left(c_{0}, c_{p}\right) \wedge A^{x_{1}}\left(c_{1}, c_{0}\right) \rightarrow P^{x_{0} 山 x_{1}}\left(c_{1}, c_{p}\right)
$$

$d_{i}$ for $0<i<p$ is induced by

$$
\mu: A^{x_{i}}\left(c_{i}, c_{i-1}\right) \wedge A^{x_{i+1}}\left(c_{i+1}, c_{i}\right) \rightarrow A^{x_{i} 山 x_{i+1}}\left(c_{i+1}, c_{i-1}\right)
$$

and $d_{p}$ is induced by

$$
r: A^{x_{p}}\left(c_{p}, c_{p-1}\right) \wedge P^{x_{0}}\left(c_{0}, c_{p}\right) \rightarrow P^{x_{p} \sqcup x_{0}}\left(c_{0}, c_{p-1}\right)
$$

Note that in the case where $P=A$, the presimplicial object $T H H(A)=T H H(A, A)$ is in fact a precyclic object via $t: \Omega^{\triangle x} V(A)(\tau x)$ induced by cyclic permutation. In the case where $A$ has a unit and $P$ is unital $T H H(A, P)$ is a simplicial object: the degeneracies are defined by using $1_{s^{\circ}}\left(i d_{c_{i}}\right)$ to insert a factor $A_{0}\left(c_{i}, c_{i}\right)$ in the $i$ th slot of $V(A, P)\left(\sigma_{i} x\right)$. Again, if $P=A$ then $T H H(A)$ is a cyclic object.

This colimit system is good in the sense of [2,1.5], i.e. thanks to the goodness of the index category $I$ and the connectivity assumptions, THH can be approximated in each degree by terms $\Omega^{U_{x}} V(A, P)(x)$ provided each coordinate in $x$ is big.

Note. In the case where the connectivity hypothesis on the ring functor and bimodule is weakened (see Section 1.1), we have to filter this object. One way to do this is
filtering by $d^{A}$ : for each $d \in N$ set $V^{d}(A, P)(x)$ to be the wedge over objects with $d^{A} \leq d$ and let

$$
T H H_{p}(A, P)=\underset{x \in I^{p+1}}{\text { holim }} \lim _{d \rightarrow \infty} \Omega^{\triangle x} V^{d}(A, P)(x) .
$$

The structure maps respect the filtering and so this defines a (pre-) simplicial object as above, with cyclic structure if $P=A$.

There are several ways of turning $T H H$ into a spectrum. The most elementary is the following. For any $X \in s_{*} E \mathscr{E}$ s let $T H H(A, P ; X)$ be defined as the diagonal of

$$
T H H_{p}\left(A, P ; X_{q}\right)=\underset{x \in P^{p+1}}{\operatorname{holim}} \Omega^{\sqcup_{x}}\left(X_{q} \wedge V(A, P)(x)\right)
$$

This defines a spectrum by setting

$$
\underline{T H H}(A, P ; X)=\left\{m \mapsto T H H\left(A, P ; S^{m} \wedge X\right)\right\} .
$$

As before, where appropriate we simplify notation to $\underline{\underline{T H}}(A, P)=\underline{\underline{T H}}\left(A, P ; S^{0}\right)$, $\underline{\underline{T H H}}(A, X)=\underline{\underline{T H H}}(A, A ; X)$ and $\underline{\underline{T H H}}(A)=\underline{\underline{T H H}}\left(A ; S^{0}\right)$. If $X$ is a cyclic space $T H H(A ; X)$ may again be considered as a (pre-) cyclic object under the diagonal action.

One should note that $T H H(A, P)$ only depends on the values of $A$ and $P$ on the objects of $\mathscr{C}$. More precisely, if $d \mathscr{C}$ is the subcategory with all objects, but only identity morphisms, then $A$ (resp. $P$ ) may be considered as a ring functor over $d \mathscr{C}$, say $d A$ (resp. $d A$-bimodule, say $d P$ ), by composition with $s_{*} \mathscr{E} n s^{\mathscr{B}^{\circ} \times \mathcal{C}} \xrightarrow{(\text { incl. })^{*}} s_{*} \mathscr{E} n s^{d 8^{\circ} \times d \boldsymbol{C}}$. Then $T H H(d A, d P)=T H H(A, P)$.

### 1.4. Calculations on examples

In the following subsections we will establish the most basic properties of the THH construction on some particularly useful examples. First we will look at its rational homotopy type. Then we will show that $T H H$ of the ring functor on the category $\mathscr{C}$ given by $X \mapsto \mathscr{C}(-,-)_{+} \wedge X$ has the stable homotopy type of the cyclic nerve of $\mathscr{C}$ (see below). Finally we will obtain a simpler description in the case where the bimodule is linear. This description will later be used in Chapter 2 to give a simple description in the case of the linear ring functor associated to an additive category.
1.4.1. The additive cyclic nerve. Recall the definition of the additive cyclic nerve of a small (simplicial) linear category $\mathfrak{C}$ with coefficients in a $\mathfrak{C}$ bimodule $\mathbf{M}: \mathbb{C}^{\circ} \otimes \mathbb{C} \rightarrow \boldsymbol{S} \mathscr{A} b$.

$$
C N_{q}(\mathbb{C}, M)=\bigoplus_{\left(c_{0}, \ldots, c_{q}\right) \in \mathscr{母}^{q+1}} M\left(c_{0}, c_{q}\right) \otimes_{Z} \mathbb{C}\left(c_{1}, c_{0}\right) \otimes_{Z} \cdots \otimes_{Z} \mathbb{C}\left(c_{q}, c_{q-1}\right)
$$

with face and degeneracy operations as for $T H H$. Note that if $M=\mathbb{C}(-,-)$ this is a cyclic object.

One example of the additive cyclic nerve is the following. Let $\mathscr{C}$ be any small category and $M: \mathscr{C}^{\circ} \times \mathscr{C} \rightarrow \mathscr{A} b$ any functor. Then $C N .(Z \mathscr{C}, M)$ is the standard resolution of Mitchell (see $[1,11])$ for the homology $\operatorname{Tor}^{\mathscr{C}^{\circ} \otimes \mathscr{C}}\left(Z_{\mathscr{C}}, M^{*}\right)$ (meaning Tor in the abelian category of all functors from $\mathscr{C}^{\circ} \times \mathscr{C}$ to $\mathscr{A} b$ of $Z \mathscr{C}(-,-)$ and $\left.M^{*}:\left(\mathscr{C}^{\circ} \times \mathscr{C}\right)^{\circ} \cong \mathscr{C}^{\circ} \times \mathscr{C} \xrightarrow{M} \mathscr{A} b\right)$. We will in a later section show that when $\mathscr{C}$ is additive this homology actually coincides with the homotopy of $T H H$ of the ring functor $X \mapsto \mathscr{C}(-,-) \otimes \boldsymbol{Z}[X] / \boldsymbol{Z}[*]$.

In this context the proposition below should be compared with the statement that topological Hochschild theory of a ring rationally coincides with ordinary Hochschild homology of the ring.

Given an $A \in \mathscr{F} \mathscr{C}$, recall from 1.2.9 the definition of $A_{\mathrm{E}}$, the "Eilenberg-MacLane" ring functor. Similarly we may for any $A$ bimodule $P$ associatc to it an $A_{\mathrm{E}}$ bimodule by

$$
P_{\mathrm{E}}^{X}(a, b)=\prod_{i=0}^{\infty} \pi_{i}(\underline{P}(a, b)) \otimes \tilde{Z}\left[S^{i} \wedge X\right]
$$

1.4.2. Definition (The linear category associated to a ring functor). Given any ring functor with unit $A$ on a category $\mathscr{C}$ we may associate to it a simplicial linear category $\mathfrak{C}_{A}$ in the following way. We let the objects be the same as for $\mathscr{C}$, but with morphism sets $\mathbb{C}_{A}(a, b)=A_{\mathrm{E}}^{0}(a, b)\left(=A_{\mathrm{E}}^{S^{0}}(a, b)\right.$, in the notation adopted in 1.1.7). Similarly if $P$ is any unital $A$ bimodule we may form a bimodule $M_{P}: \mathbb{C}_{A}^{\circ} \otimes \mathbb{C} \rightarrow s \mathscr{A} b$ given by $M_{P}(a, b)=P_{E}^{0}(a, b)$.

There is a map from $\operatorname{THH}\left(A_{\mathrm{E}}, P_{\mathrm{E}}\right)$ to $C N\left(\mathbb{C}_{A}, M_{P}\right)$ given as follows: Send smashes of simplicial abelian groups to tensors, and send the wedge to the sum.


As $\underline{\underline{A}}_{\mathrm{E}}$ and $\underline{\underline{P}}_{\mathrm{E}}$ are Eilenberg-MacLane spectra, the latter is isomorphic to

$$
\underset{x \in I^{0+1}}{\text { holim }} \Omega^{\sqcup x} \tilde{Z}\left[S^{\sqcup x}\right] \otimes \underset{\left(c_{0}, \ldots, c_{q}\right) \in \mathscr{\zeta}^{q+1}}{ } P_{\mathrm{E}}^{0}\left(c_{0}, c_{q}\right) \otimes A_{\mathrm{E}}^{0}\left(c_{1}, c_{0}\right) \otimes \cdots \otimes A_{\mathrm{E}}^{0}\left(c_{q}, c_{q-1}\right)
$$

which again maps by a homotopy equivalence to

$$
\bigoplus_{\left(c_{0}, \ldots, c_{q}\right) \in \mathscr{母}^{q+1}} P_{\mathrm{E}}^{0}\left(c_{0}, c_{q}\right) \otimes A_{\mathrm{E}}^{0}\left(c_{1}, c_{0}\right) \otimes \cdots \otimes A_{\mathrm{E}}^{0}\left(c_{q}, c_{q-1}\right)=C N_{q}\left(\mathcal{C}_{A}, M_{P}\right)
$$

This map is compatible with the simplicial structure. In the case $P=A$ it is compatible with the cyclic structure as well.
1.4.3. Proposition (Rational computation of $T H H$ ). Let $A$ be a ring functor with unit on $\mathscr{C}$ and $P$ a unital $A$ bimodule, and let $\mathbb{C}_{A}$ and $M_{P}$ be associated linear category and bimodule. Then $\boldsymbol{Q}_{\infty} T H H(A, P) \rightarrow \boldsymbol{Q}_{\infty} C N\left(\mathbb{C}_{A}, M_{P}\right)$ is a homotopy equivalence.

Proof. Recall from 1.2.8 that given any $A \in \mathscr{F} \mathscr{C}$ we associated to it a stably equivalent $A_{\Omega} / \mathscr{F} \mathscr{C}$ whose spectrum was actually an $\Omega$-spectrum. By the same process we may associate an $A_{\Omega}$ bimodule $P_{\Omega}$ to any $A$ bimodule $P$. In Section 1.6 .10 we will show that stably equivalent ring functors and bimodules have equivalent $T H H \mathrm{~s}$, so $T H H(A, P) \stackrel{\simeq}{\Rightarrow} T H H\left(A_{s}, P_{S}\right)$. Now, the rationalization of any $H$-space has vanishing Postnikov invariants, so we get homotopy equivalences $\boldsymbol{Q}_{\infty} A_{\Omega}^{X}(a, b) \xrightarrow{\simeq} \boldsymbol{Q}_{\infty} A_{\mathrm{E}}^{X}(a, b)$ and $\boldsymbol{Q}_{\infty} P_{\Omega}^{X}(a, b) \xrightarrow{\simeq} \boldsymbol{Q}_{\infty} P_{E}^{X}(a, b)$ where, using the notation of [5], $\boldsymbol{Q}_{\infty}$ means rationalization. Rationalization commutes with smash, infinite wedges of simply connected spaces and loops of fibrant nilpotent spaces by [5, Ch. V, 4.6 and 5.1], and for any bisimplicial set of horizontal rationalization induces a rationalization of the diagonal. By Bökstedt's approximation lemma [2, 1.5] this means that

$$
\boldsymbol{Q}_{\infty} T H H(A, P) \xrightarrow{\simeq} Q_{\infty} T H H\left(A_{\mathrm{E}}, P_{\mathrm{E}}\right) .
$$

So to prove the proposition it is enough to show that

$$
T H H\left(A_{\mathrm{E}}, P_{\mathrm{E}}\right) \rightarrow C N\left(\mathbb{C}_{A}, M_{P}\right)
$$

is a rational homotopy equivalence. Replace $A_{\mathrm{E}}$ and $P_{\mathrm{E}}$ by the stably equivalent ring functor and bimodule given by replacing the product by the wedge. Then $T H H_{p}\left(A_{\mathrm{E}}, P_{\mathrm{E}}\right)$ is homotopy equivalent to holim $\Omega^{\mathrm{U}^{\triangle}} W(x)$ where $W(x)$ denotes

$$
\bigvee_{\left(c_{0}, \ldots, c_{p}\right) \in 母^{p+1}} \bigvee_{\left(i_{0}, \ldots, i_{p}\right) \in N_{0}^{p+1}} \pi_{i_{0}}\left(\underline{P}\left(c_{0}, c_{p}\right)\right) \otimes \tilde{Z}\left[S^{i_{0}+x_{0}}\right]
$$

$$
\wedge \pi_{i_{1}}\left(\underline{\underline{A}}\left(c_{1}, c_{0}\right)\right) \otimes \tilde{Z}\left[S^{i_{1}+x_{1}}\right] \wedge \cdots \wedge \pi_{i_{p}}\left(\underline{\underline{A}}\left(c_{p}, c_{p-1}\right)\right) \otimes \tilde{Z}\left[S^{i_{p}+x_{p}}\right]
$$

where $N_{0}$ denotes the set of natural numbers including zero. For each $\left(c_{0}, \ldots, c_{n}\right) \in \mathscr{C}^{n+1},\left(i_{0}, \ldots, i_{p}\right) \in N_{0}^{p+1}$ and $x=\left(x_{0}, \ldots, x_{n}\right) \in I^{n+1}$ consider the simplicial sets

$$
\begin{aligned}
& X= \pi_{i_{0}}\left(\underline{\underline{P}}\left(c_{0}, c_{p}\right)\right) \otimes \tilde{Z}\left[S^{i_{0}+x_{0}}\right] \wedge \pi_{i_{1}}\left(\underline{\underline{A}}\left(c_{1}, c_{0}\right)\right) \otimes \tilde{Z}\left[S^{i_{1}+x_{1}}\right] \\
& \wedge \cdots \wedge \pi_{i_{p}}\left(\underline{\underline{A}}\left(c_{p}, c_{p-1}\right)\right) \otimes \tilde{Z}\left[S^{i_{p}+x_{p}}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
& Y= \pi_{i_{0}}\left(\underline{\underline{P}}\left(c_{0}, c_{p}\right)\right) \otimes \tilde{Z}\left[S^{i_{0}+x_{0}}\right] \otimes \pi_{i_{1}}\left(\underline{\underline{A}}\left(c_{1}, c_{0}\right)\right) \otimes \tilde{Z}\left[S^{i_{1}+x_{1}}\right] \\
& \otimes \cdots \otimes \pi_{i_{p}}\left(\underline{A}\left(c_{p}, c_{p-1}\right)\right) \otimes \tilde{Z}\left[S^{i_{p}+x_{p}}\right]
\end{aligned}
$$

Both spaces have the rational homology groups

$$
H_{q}(; \boldsymbol{Q})= \begin{cases}0 & \text { if } 0<q<\Sigma \\ \pi_{i_{0}}\left(\underline{\underline{P}}\left(c_{0}, c_{p}\right)\right) \otimes \cdots \otimes \pi_{i_{p}}\left(\underline{\underline{A}}\left(c_{p}, c_{p-1}\right)\right) \otimes \boldsymbol{Q} & \text { if } q=\Sigma \\ 0 & \text { if } \Sigma<q<2 \Sigma\end{cases}
$$

where $\Sigma=\sum_{k=0}^{p}\left(i_{k}+x_{k}\right)$. The map
is $2((\cup \boldsymbol{x}))-1$ connected by an argument considering finite index sets and using Blakers-Massey, and finally taking the limit. Consequently the map

$$
\bigvee_{\left(c_{0}, \ldots, c_{n}\right) \in \mathscr{Y}^{n+1}} \bigvee_{\left(i_{0}, \ldots, i_{p}\right) \in N_{0}^{p+1}} X \rightarrow \bigoplus_{\left(c_{0}, \ldots, c_{n}\right) \in \mathscr{Y}^{+1}} \bigoplus_{\left(i_{0}, \ldots, i_{p}\right) \in N_{0}^{p+1}} Y \cong C N_{p}\left(\mathbb{C}_{A}, M_{P}\right) \otimes_{Z} \tilde{Z}\left[S^{\sqcup x}\right]
$$

induces an isomorphism on $H_{q}(-, \boldsymbol{Q})$ for $q<2(\sqcup \boldsymbol{x})$. Looping $\sqcup \boldsymbol{x}$ times we get that

$$
\boldsymbol{Q}_{\infty} \Omega^{\sqcup_{x}} V\left(A_{\mathrm{E}}, P_{\mathrm{E}}\right)(\boldsymbol{x}) \rightarrow C N_{n}\left(\mathfrak{C}_{\boldsymbol{A}}, M_{\boldsymbol{P}}\right) \otimes_{Z} \boldsymbol{Q}
$$

is $(\sqcup \boldsymbol{x})-1$ connected for every $\boldsymbol{x}$. The result then follows.
1.4.4. The topological Hochschild homology of $X \mapsto \mathscr{C}(-,-)_{+} \wedge X$ is stably the cyclic nerve of $\mathscr{C}$. As a simple extension of the computation of $T H H$ of a monoid we consider $T H H$ of the half smash ring functor of Example 1.2.2.
1.4.5. Definition (The cyclic nerve of a category). Given a category $\mathscr{C}$, the cyclic nerve of $\mathscr{C}$ is the simplicial object $N^{\text {cy }}(\mathscr{C})$ given by

$$
N_{p}^{c y}(\mathscr{C})=\left\{\text { circular diagrams } c_{p} \leftarrow c_{0} \leftarrow \cdots \leftarrow c_{p-1} \leftarrow c_{p} \text { in } \mathscr{C}\right\}
$$

with face, degeneracies and cyclic operator as in Hochschild homology.

For any (simplicial) category $\mathscr{C}$ let us call the ring functor given by $X \mapsto \mathscr{C}(-,-)_{+} \wedge X$ simply $\mathscr{C}_{\propto_{\propto}}$. Furthermore for any space $X$ we let $Q(X)=$ $\lim _{k \rightarrow \infty} \Omega^{k}\left(S^{k} \wedge X\right)$.
1.4.6. Proposition. Let $\mathscr{C}$ be any simplicial category. Then

$$
T H H\left(\mathscr{C}_{\infty}\right) \simeq Q\left(N^{\mathrm{cy}}(\mathscr{C})_{+}\right)
$$

Proof. Consider the bisimplicial object $X_{* *}$ given by

$$
X_{p q}=\underset{x \in I^{p+1}}{\operatorname{holim}} \Omega^{\mathrm{x}}\left(S^{\breve{ } x} \wedge\left(N_{q}^{\text {cy }}(\mathscr{C})\right)_{+}\right)
$$

$T H H\left(C_{\infty}\right)$ is isomorphic to the diagonal diag $X$ by the following rewriting：

$$
\begin{aligned}
& T H H_{p}\left(\mathscr{C}_{\infty}\right) \cong \underset{x \in I^{p+1}}{\operatorname{holim}} \Omega_{\left(c_{0}, \ldots, c_{p}\right) \in \mathscr{Q}^{p+1}}^{\mathrm{Lx}_{x}} \mathscr{C}\left(c_{0}, c_{p}\right)_{+} \wedge \mathscr{C}\left(c_{1}, c_{0}\right)_{+} \\
& \wedge \cdots \wedge \mathscr{C}\left(c_{p}, c_{p-1}\right)_{+} \wedge S^{\Delta x} \\
& \cong \underset{x \in I^{p+1}}{\operatorname{holim}} \Omega^{\Delta_{x}} S^{山_{x}} \underset{\left(c_{0}, \ldots, c_{p}\right) \in \mathscr{Q}^{++1}}{ }\left(\mathscr{C}\left(c_{0}, c_{p}\right) \times \mathscr{C}\left(c_{1}, c_{0}\right)\right. \\
& \left.\times \cdots \times \mathscr{C}\left(c_{p}, c_{p-1}\right)\right)_{+} \\
& =\xrightarrow[x \in I^{p+1}]{\text { holim }} \Omega^{山_{x}} S^{\Delta x} \wedge\left(\underset{\left(c_{0}, \ldots, c_{p}\right) \in \mathscr{Q}^{p+1}}{ } \mathscr{C}\left(c_{0}, c_{p}\right) \times \mathscr{C}\left(c_{1}, c_{0}\right)\right. \\
& \left.\times \cdots \times \mathscr{C}\left(c_{p}, c_{p-1}\right)\right)_{+} \\
& =\underset{x \in I^{p+1}}{\operatorname{holim}} \Omega^{\Delta x} S^{\Delta x} \wedge\left(N_{p}^{\text {cy }}(\mathscr{C})\right)_{+} .
\end{aligned}
$$

The horizontal face maps $X_{p . *} \rightarrow X_{p-1, *}$ are all homotopy equivalence，and so the inclusion $Q\left(N^{\text {cy }}(\mathscr{C})_{+}\right)=X_{0, *} \subseteq X_{* *}$ is also a homotopy equivalences．

Of course this could be generalized to a larger class of ring functors and bimodules， in the same vein as the result above．

1．4．7．$T H H$ with coefficients in bilinear bimodules may be modelled as a simplicial abelian group．Recall that if $Y$ is a simplicial abelian group we may define the loop space on $Y$ to be the simplicial abelian group $\Omega Y=s_{*} \mathscr{E} n s\left(S^{1}, Y\right)$ where $s_{*} \mathscr{E} n s(-, Y)$ is given the usual abelian group structure．Furthermore，we may for any functor from a small category to the category of simplicial abelian groups，say $Y: \mathscr{C} \rightarrow s \mathscr{A} b$ ，define the homotopy colimit holim $\underset{\mathscr{C}}{ } Y$ by using sums instead of wedges in the simplicial replacement lemma（see［5，Chapter XII，Section 5］）．

Let $A$ be a ring functor on $\mathscr{C}$ ，and let $P$ be a bilinear $A$－module．More precisely：we assume that $P \in L \mathscr{C}$ factors through

$$
s \mathscr{A} b^{6^{\circ} \times \varnothing} \rightarrow s_{*} \mathscr{E} n s^{6^{\circ} \times \varnothing}
$$

and that the action of $A$ is distributive over addition，i．e．the two obvious maps

$$
\left(P^{X}(a, b) \times P^{X}(a, b)\right) \wedge A^{Y}(c, a) \rightarrow P^{X \wedge Y}(c, b)
$$

are equal．Then we can rewrite $\operatorname{THH}(A, P)$ in the following fashion．
1.4.8. Proposition. Let $A$ and $P$ be as above, then $T H H(A, P)$ is weakly equivalent to the (pre-) simplicial abelian group

$$
\begin{aligned}
& \left\{p \mapsto \underset{x \in I^{p^{+1}}}{\operatorname{holim}} \Omega_{\left(c_{0}, \ldots, c_{p}\right) \in \mathscr{C}^{p+1}}^{\sqcup_{x}} P^{x_{0}}\left(c_{0}, c_{p}\right) \otimes \tilde{\boldsymbol{Z}}\left[A^{x_{1}}\left(c_{1}, c_{0}\right)\right]\right. \\
& \left.\otimes \cdots \otimes \tilde{\boldsymbol{Z}}\left[A^{x_{p}}\left(c_{p}, c_{p-1}\right)\right]\right\}
\end{aligned}
$$

Proof. By Lemma 1.6.9 we may assume that $\mathscr{C}$ only has a finite number of objects, and hence it is possible to choose a common constant $c$ for the connectivity of the structure maps for $A$ and $P$ (see Definition 1.1.1(2)).

Let $\boldsymbol{x}=\left(x_{0}, \ldots, x_{p}\right) \in I^{p+1}$. Then by the Freudenthal suspension theorem

is $(\sqcup \boldsymbol{x})-2 p-1$ connected. Now, by the requirement on $P$ as an element of $L \mathscr{C}$

$$
S^{k} \wedge P^{n}\left(c_{0}, c_{p}\right) \cong S^{k-1} \wedge\left(S^{1} \wedge P^{n}\left(c_{0}, c_{p}\right)\right) \rightarrow S^{k-1} \wedge P^{n+1}\left(c_{0}, c_{p}\right)
$$

is $k+2 n-3-c$ connected, and so $S^{N} \wedge P^{x_{0}}\left(c_{0}, c_{p}\right) \rightarrow P^{x_{0}+N}\left(c_{0}, c_{p}\right)$ is $2 x_{0}+N-$ $3-c$ connected. Thus

is $x_{0}+(\sqcup \boldsymbol{x})-p-3-c$ connected. But the latter formula is, by the description of homology by means of spectra, just the reduced homology of $A^{x_{1}}\left(c_{1}, c_{0}\right) \wedge \cdots \wedge$ $A^{x_{p}}\left(c_{p}, c_{p-1}\right)$ with coefficients in $P^{x_{0}}\left(c_{0}, c_{p}\right)$. This is hence weakly equivalent to

$$
P^{x_{0}}\left(c_{0}, c_{p}\right) \otimes \tilde{Z}\left[A^{x_{1}}\left(c_{1}, c_{0}\right) \wedge \cdots \wedge A^{x_{p}}\left(c_{p}, c_{p-1}\right)\right]
$$

Now, this is $\sqcup \boldsymbol{x}-1-p$ connected, so by Blakers-Massey

$$
\begin{aligned}
& \bigvee_{\left(c_{0}, \ldots, c_{p}\right) \in \mathscr{母}^{p+1}} P^{x_{0}}\left(c_{0}, c_{p}\right) \otimes \tilde{Z}\left\lceil A^{x_{1}}\left(c_{1}, c_{0}\right) \wedge \cdots \wedge A^{x_{p}}\left(c_{p}, c_{p-1}\right)\right] \\
& \quad \subseteq \bigoplus_{\left(c_{0}, \ldots, c_{p}\right) \in \mathscr{母}^{p+1}} P^{x_{0}}\left(c_{0}, c_{p}\right) \otimes \tilde{Z}\left[A^{x_{1}}\left(c_{1}, c_{0}\right) \wedge \cdots \wedge A^{x_{p}}\left(c_{p}, c_{p-1}\right)\right]
\end{aligned}
$$

is $2\left(\sqcup_{\boldsymbol{x}}\right)-2 p-3$ connected.

Collecting all maps we get that

is $\min \left(x_{0}-p-3-c,(\sqcup x)-2 p-3\right)$ connected (where we have used the identity $\tilde{Z}[X \wedge Y] \cong \tilde{Z}[X] \otimes \tilde{Z}[Y]$ in the last statement, and again loops of simplicial abelian groups are chosen as above). The map on homotopy colimits (as sets) is consequently a weak equivalence. Note that in representing the latter homotopy colimit it does not matter, up to weak equivalence, whether we use wedges or sums in the simplicial replacement functor. This follows by Blakers-Massey and the fact that the connectivity increases with the cardinality of $\sqcup \boldsymbol{x}$. Letting $p$ vary and going to the diagonal we get the stated result.

### 1.5. Cyclic structure and topological cyclic homology

The examples of calculations for $T H H$ above are somewhat deceiving, since a major importance of the THH construction lies in the fact that we obtain a cyclic structure when the ring functor itself serves as the bimodule. The computations in 1.4.3, 1.4.6 and 1.4.8 do not preserve this cyclic structure, but in general we will try to keep this structure as long as possible. Going carefully through the proofs in the rational and the $X \mapsto \mathscr{C}(-,-)_{+} \wedge X$ examples we may in fact recover the cyclic structure. We will not do so here, but for general computations it will be essential that we carry with us this information.
1.5.1. Definition. Let $s d_{r}$ be the edgewise subdivision functor (see [3] or 1.5.5), and note that the definition extends to precyclic objects to give a precyclic object with a natural $C_{r}$ action. We say that a precyclic map $f: X \rightarrow Y$ is a $C$-equivalence if $s d_{r} f^{C_{r}}$ is a weak equivalence for all $r$.

If $f$, in addition to being a $C$-equivalence, is actually a cyclic map, this implies that $|f|$ is a $C_{r}$ equivariant homotopy equivalence for every $r$. This is true since for all $C_{s} \subseteq C_{r}$

commutes giving that $|f|^{C_{s}}$ is a homotopy equivalence, and so $|f|$ is a $C_{r}$ equivariant homotopy equivalence by the equivariant Whitehead theorem [16].

Let $A$ be a ring functor on a small category $\mathscr{C}$ and let $r \in N$. In dimension $q-1$ we have

$$
\begin{aligned}
\left(s d_{r} T H H(A)^{c_{r}}\right)_{q-1} & =\left(T H H_{r q-1}(A)\right)^{c_{r}}=\left(\underset{x \in I^{r}}{\text { holim }} \Omega^{\Delta x} V(A)(x)\right)^{c_{r}} \\
& =\underset{x \in I^{4}}{\operatorname{holim}}\left(\Omega^{\Delta\left(x^{\sqcup r}\right)} V(A)\left(x^{\text {Ur }}\right)\right)^{c_{r}}
\end{aligned}
$$

where $\boldsymbol{x}^{\mathrm{Ur}} \in I^{q r}$ is the image of the $r$-fold diagonal of $\boldsymbol{x} \in I^{q}$ (which are the only fixed points under the $C_{r}$ action on $I^{r q}$ ).

Note that $\left|\left(S^{x_{1}} \wedge \cdots \wedge S^{x_{q}}\right)^{\wedge r}\right|^{C_{r}} \cong\left|S^{x_{1}} \wedge \cdots \wedge S^{x_{q}}\right|$ and $\left|V(A)\left(x^{{ }^{4} r}\right)\right|^{C_{r}} \cong|V(A)(x)|$. By restricting the $C_{r}$ equivariant maps from $\left.\mid\left(S^{x_{1}} \wedge \ldots \wedge S^{x_{q}}\right)^{\wedge}\right) \mid$ to $\left|V(A)\left(x^{\Delta r}\right)\right|$ to the fixed point sets of the $C_{r}$ action we get a map to

$$
\xrightarrow[x \in I^{q}]{\text { holim }} \Omega^{x} V(A)(x)=T H H_{q-1} A
$$

This map extends to a simplicial map denoted $\phi_{r}$. Likewise we may define a map

$$
\left(s d_{s r} T H H(A)\right)^{C_{s r}} \rightarrow\left(s d_{s} T H H(A)\right)^{c_{s}}
$$

also denoted $\phi_{r}$.
1.5.2. Definition. The map constructed above is the Frobenius map

$$
\phi_{r}: s d_{s r} T H H(A)^{c_{s r}} \rightarrow s d_{s} T H H(A)^{C_{s}} .
$$

Note that in view of the results in [9], the map $\phi_{r}$ should definitely not be called the Frobenius map. In fact, the inclusion of fixed points is very much closer to the classical Frobenius map on the Witt vectors. However, in this paper we will stick to the notation in [3] etc.

Let $\mathscr{F}$ be the category with objects the natural numbers, and a morphism $f_{p, q}: p a q \rightarrow a$ for every triple $a, p, q \in N$ subject to $f_{p, q} \circ f_{r, s}=f_{p r, q s}$. If $A$ is a unital ring functor there is a functor $T$ from $\mathscr{F}$ to spaces sending $a \in o b \mathscr{F}$ to $\left|s d_{a} T H H(A)^{C_{a}}\right|$ and $f_{p, q}$ to $\left|\phi_{p}\right|^{\circ} i_{q}$ where $i_{q}:\left|s d_{a q} T H H(A)^{C_{a q}}\right| \cong|T H H(A)|^{C_{a q}} \subseteq|T H H(A)|^{C_{a}} \cong$ $\left|s d_{a} T H H(A)^{C_{a}}\right|$ is the inclusion.
1.5.3. Definition. Let $A$ be a unital ring functor and $T: \mathscr{F} \rightarrow$ spaces the functor above. Then

$$
T C(A)=\underset{\mathscr{F}}{\underset{\mathrm{holim}}{ }} T
$$

is called the topological cyclic homology of $A$.

A map $A \rightarrow B$ of unital ring functors which gives a $C$-equivalence $T H H(A) \rightarrow T H H(B)$ clearly induces a homotopy equivalence $T C(A) \rightarrow T C(B)$. This is one of the motivations for the care taken with the cyclic structure in the next section. Another approach would be that of Goodwillie [8] who characterized the fibre of the Frobenius map $s d_{s r} T H H(A)^{C_{s r}} \rightarrow s d_{s} T H H(A)^{C_{s}}$ as $\lim _{m \rightarrow \infty} \Omega^{m}\left(T H H\left(A ; S^{m}\right)_{h c_{s}}\right)$ up to natural equivalence. His proof passes over to our case by replacing $W=$ $X \wedge\left(A\left(S^{x_{0}}\right) \wedge \cdots \wedge A\left(S^{x_{i}}\right)\right)^{\wedge s r}$ with $X \wedge V(A)\left(x^{u r s}\right)$ everywhere. Thus we get that any $\operatorname{map} A \rightarrow B$ inducing a weak equivalence $\operatorname{THH}(A ; X) \rightarrow \operatorname{THH}(B ; X)$ for every finite pointed simplicial set $X$ induces a weak equivalence on $T C$.

However, in proving the theorems we are concerned with in this chapter, giving the appropriate weak equivalences is often not much simpler than giving the $C$-equivalences, and we will stick to this more direct route. We conclude this section with some notions useful at that end. References for unproven statements are [14] for general presimplicial properties, [10] appendix for precyclic properties, and [3] for subdivision and general cyclic properties.
1.5.4. Presimplicial sets. Given two presimplicial maps $f, g: X \rightarrow Y$ we say that they are prehomotopic (earlier: "semi-homotopic") if there exists a presimplicial map $H: H \times \Delta(1) \rightarrow Y$ such that $f(x)=H(x,(1, \ldots, 1))$ and $g(x)=H(x,(0, \ldots, 0))$ for any $x \in X_{q}$ where $(1, \ldots, 1)$ and $(0, \ldots, 0)$ in $\Delta(1)_{q}=\Delta([q],[1])$ are the maps sending everything to 1 and 0 respectively. Similarly a prehomotopy equivalence is a presimplicial map $f: X \rightarrow Y$ such that there exists a map $g: Y \rightarrow X$ such that both the composites $g f$ and $f g$ are prehomotopic to the identity. The realization of a presimplicial set $X$ is defined just as for a simplicial set:

$$
\|X\|=\coprod_{q}|\Delta(q)| \times X_{q} /\left(\theta_{*} \sigma, x\right) \sim\left(\sigma, \theta^{*} x\right)
$$

for $\theta \in \Delta_{m}([p],[q]), \sigma \in|\Delta(p)|$ and $x \in X_{q}$.
For simplicial spaces this is the same as the thick realization of Segal [15]. Recall the construction $X \mapsto \tilde{X}$ for adjoining degeneracies to a simplicial set described in 1.3.1. The spaces $|\tilde{X}|$ and $\|X\|$ are homeomorphic, and if $X$ already is a simplicial set then the canonical map $\tilde{X} \rightarrow X$ is a weak equivalence. We may even display a section: if $x \in X_{q}$ there is a unique surjective map $\phi \in \Delta([q],[p])$ and $y \in X_{p}$ such that $x=X \circ \phi(y)$. We then send $x$ to $y$ in the $\phi$ th coordinate. We say that two presimplicial maps are weakly homotopic if their realizations are homotopic, and that a map is a weak equivalence if the realization is a homotopy equivalence. If $X$ is a simplicial set or a Kan presimplicial set [14] then two maps $X \rightarrow Y$ are weakly homotopic if they are prehomotopic, and if $Y$ is a Kan as well these notions coincide.

Let $G$ be a group and $X$ a presimplicial set on which $G$ acts presimplicially. Then $\tilde{X}$ naturally becomes a simplicial $G$ set. If $X$ already were a simplicial $G$ set then $\tilde{X} \rightarrow X$ induces a $G$ equivariant homotopy equivalence $|\tilde{X}| \rightarrow|X|$.
1.5.5. Edgewise subdivision. Let $r \in N$. Consider the functor $s d_{r}: \Delta \rightarrow \Delta$ given by sending an object [q] to

$$
\overbrace{[q] \sqcup[q] \sqcup \ldots \sqcup[q]}^{r \text { times }}=[(q+1) r-1]
$$

and a morphism $\phi:[q] \rightarrow[p]$ to

$$
\phi \sqcup \ldots \sqcup \phi:[q] \sqcup \ldots \sqcup[q] \rightarrow[p] \sqcup \ldots \sqcup[p]
$$

(i.e. $\operatorname{sd} d_{r} \phi(a(q+1)+b)=a(p+1)+\phi(b)$ for $0 \leq a<r$ and $\left.0 \leq b \leq q\right)$. If $X$ is a simplicial object we define $s d_{r} X$ to be $X \circ s d_{r}$. If $X$ is a simplicial set there is a homeomorphism of realizations $D_{r}:\left|s d_{r} X\right| \xrightarrow{\cong}|X|$ (see [3]). Note that $s d_{r}$ restricts to a functor $\Delta_{m} \rightarrow \Delta_{m}$, and so can we define $s d_{r} X$ for any presimplicial object.

The realization of a cyclic set comes naturally with a circle action, and we will be concerned with the action of the finite subgroups. Using the edgewise subdivision, this action can be described simplicially. Let $C_{r} \subseteq S^{1}$ denote the subgroup of the circle with $r$ elements. Given a (pre-) cyclic set $X$, the $r$ th edgewise subdivision has a (pre-) simplicial $C_{r}$ action given by $t_{(p+1) r-1}^{p+1}: X_{(p+1) r-1}=\left(s d_{r} X\right)_{p} \rightarrow\left(s d_{r} X\right)_{p}$. In the cyclic situation we get that $\left|s d_{r} X^{C_{r}}\right| \cong|X|^{C_{r}}$ where $X$ is some cyclic set.
1.5.6. Special homotopies. Let $\mathscr{\mathscr { F }}$ be the groupoid on two objects, i.e., is the category with two objects, say 0 and 1 , and two non-identity isomorphisms $0 \rightarrow 1$ and $1 \rightarrow 0$. Recall that the cyclic nerve of a category $\mathscr{C}$ is a cyclic set, and that the $C_{r}$ fixed points $s d_{r} N^{c y}(\mathscr{C})^{C_{r}}$ can naturally be identified with $N^{c y}(\mathscr{C})$. This fact, together with the fact that $\Delta(1)=N(0 \rightarrow 1)$ is a subset of $N^{\text {cy }}(\mathscr{F})$ makes the following definition useful.
1.5.7. Definition (Special homotopies [10]). Let $f, g: X \rightarrow Y$ be two precyclic maps. We say that $f$ and $g$ are specially homotopic if there is a precyclic map

$$
H: X \times N^{c y}(\mathscr{I}) \rightarrow Y
$$

such that $f(x)=H(x,(1=1=\cdots=1))$ and $g(x)=H(x,(0=0=\cdots=0))$ for any $x \in X_{q}$. We will say that $f$ is a special homotopy equivalence if there exists a precyclic $\operatorname{map} \bar{f} ; Y \rightarrow X$ such that both $f \circ \bar{f}$ and $\bar{f}$ are specially homotopic to the identity. In this case we will say that $X$ and $Y$ are specially homotopy equivalent, and that $\bar{f}$ is a special homotopy inverse.
1.5.8. Lemma (McCarthy [10]). If $f, g: X \rightarrow Y$ are specially homotopic maps and $X$ a cyclic set, then for all $r>0, s d_{r} f^{c_{r}}$ and $s d_{r} g^{c_{r}}: s d_{r} X^{c_{r}} \rightarrow s d_{r} Y^{c_{r}}$ are prehomotopic (and hence weakly homotopic). If f has a special homotopy inverse and both $X$ and $Y$ are cyclic sets, then $s d_{r} f^{C r}$ is a prehomotopy equivalence, and thus $f$ is a C-equivalence.

Proof. We have by the observation $s d_{r} N^{c y}(\mathscr{I})^{c_{r}}=N^{c y}(\mathscr{I})$ above, that

$$
s d_{r}\left(X \times N^{c y}(\mathscr{I})\right)^{c_{r}}=s d_{r} X^{c_{r}} \times N^{c y}(\mathscr{I})
$$

and so by restriction to $\Delta(1)$ we get a prehomotopy

$$
s d_{r} X^{c_{r} \times \Delta(1) \rightarrow s d_{r} X^{c_{r}}, ~}
$$

from $s d_{r} f^{c_{r}}$ to $s d_{r} g^{c}$. The last statement now follows.
1.5.9. Presimplicial simplicial sets. Quite often the lack of degeneracies in a presimplicial simplicial set causes no trouble if we have degreewise equivalences to a bisimplicial set. Below follow some useful lemmas. The last is especially important as it tells us how to translate special homotopy equivalences of precyclic simplicial sets to $C$ equivalences of cyclic simplicial sets.
1.5.10. Lemma. If $X \rightarrow Y \in s_{*} \mathscr{E} n s^{\Delta_{n}^{o}}$ is a map of presimplicial pointed simplicial sets such that $X_{q} \xrightarrow{\sim} Y_{q} \in s_{*} \mathscr{E} n$ s is a weak equivalence for every $q$ then $X \rightarrow Y$ is a weak equivalence.

Proof. Note that even if we adjoin degeneracies to both $X$ and $Y$ we still have a degreewise weak equivalence

$$
\tilde{X}_{q}=\prod_{\substack{p \rightarrow q \\ \text { surjective } \phi \in \Delta([q],[p])}} X_{p} \xrightarrow{\simeq} \prod_{\substack{p \leq q \\ \text { sur jective } \phi \in \Delta([q],[p])}} Y_{p}=\tilde{Y}_{p}
$$

This means (see [5] XII, 4.2]) that diag $\tilde{X} \rightarrow \operatorname{diag} \tilde{Y}$ (diagonals of bisimplicial sets) is a weak equivalence and so $|\tilde{X}| \xrightarrow{\simeq}|\tilde{Y}|$ is a homotopy equivalence.
1.5.11. Lemma. Let $f, g: V \rightarrow W \in s_{*} \mathscr{E} n s^{\Delta_{m}^{0}}$ be two prehomotopic maps of presimplicial objects. Assume there is a degreewise weak equivalence $h: V \rightarrow X$ where $X$ is a simplicial object (bisimplicial set). Then $|\tilde{f}|$ and $|\tilde{g}|$ are homotopic.

Proof. Let $H: V \times \Delta[1] \rightarrow W$ be the prehomotopy. Adjoining degeneracies this is a map $\tilde{H}: \widetilde{V \times \Delta}[1] \rightarrow \tilde{W}$ which upon restriction to $\tilde{V} \vee \tilde{V}=\widetilde{V V V} \subseteq \overparen{V \times \Delta}[1]$ is simply $\tilde{f} \vee \tilde{g}$. As $h: V_{q} \rightarrow X_{q}$ is a weak equivalence for each $q$, we get that $h \times i d: V_{q} \times \Delta([q],[1]) \rightarrow X_{q} \times \Delta([q],[1])$ is a weak equivalence, and so by Lemma 1.5.10 we get that

$$
\overparen{V \times \Delta}[1] \xrightarrow{\widetilde{h \times \text { id }}} \overparen{X \times \Delta}[1] \rightarrow X \times \Delta[1] \rightarrow \tilde{X} \times \Delta[1] \leftarrow \tilde{V} \times \Delta[1]
$$

is a weak equivalence. Let $|\widetilde{V \times \Delta}[1]| \xrightarrow{\cong}|\tilde{V}| \times|\Delta[1]|$ be any map representing this weak equivalence. There is a lifting of this map which upon restriction to $|\tilde{V}| \vee|\tilde{V}| \mapsto|\tilde{V}| \times|\Delta[1]|$ is the inclusion (factor the map into a trivial cofibration followed by a trivial fibration. The cofibration splits, and as $|\tilde{V}| \vee|\tilde{V}| \mapsto|\tilde{V}| \times|\Delta[1]|$ is a cofibration we may lift past the trivial fibration). Hence this lifting composed with $|\tilde{H}|$ is the desired homotopy.
1.5.12. Proposition. Assume that we are in the situation depicted in the diagram in $s_{*} \mathscr{E} n s^{\Lambda_{m}^{0}}$ below

where $X \xrightarrow{f} Y$ is uctually a cyclic pointed simplicial set map.
Suppose for all $r, q>0$ the vertical maps induced degreewise weak equivalences $V_{r q-1}^{c_{r}} \rightarrow X_{r q-1}^{c_{r}}$ and $W_{r q-1}^{c_{r}} \rightarrow Y_{r q-1}^{c_{r}}$ and that the bottom horizontal map is a special homotopy equivalence. Then $f$ is a C-equivalence, and so for all $r$, the realization $|f|$ is a $C_{r}$ equivariant homotopy equivalence.

Proof. The hypotheses assure that $\left(s d_{r} V^{C_{r}}\right)_{q-1}=V_{r q-1}^{c_{r}} \xlongequal{\rightrightarrows} X_{r q-1}^{c_{r}}=\left(s d_{r} X^{c_{r}}\right)_{q-1}$ is a weak equivalence. In particular, by the lemma above, it shows that any prehomotopy $s d_{r} V^{C_{r}} \times \Delta(1) \rightarrow s d_{r} V^{C_{r}}$ induces a homotopy on realization. The same considerations apply to $W$ and $Y$. If $\bar{g}$ is the special homotopy inverse to $g$ let $H: V \times N^{c y}(\mathscr{I}) \rightarrow V$ be the special homotopy form the identity to $\bar{g} \circ g$. Then $s d_{r}\left(V \times N^{c y}(\mathscr{F})\right)^{c_{r}}=$ $s d_{r} V^{C_{r}} \times N^{c y}(\mathscr{I})$, and so $s d_{r} H^{c_{r}}$ is a special homotopy (and in particular a prehomotopy) from the identity to $s d_{\mathbf{r}}\left(\bar{g}^{\circ} g\right)^{c_{r}}$. Likewise for the other composition. By Lemma 1.5.11 we have that $s d_{r} V^{c C_{r}} \rightarrow s d_{r} W^{C_{r}}$ is a weak equivalence for all $r$ and hence we are done.

### 1.6. Basic properties

In this section we will establish some basic properties for the THH construction of Section 1.3. We first will show that $T H H$ behaves well under natural isomorphisms of the underlying category, and that it commutes with the direct limit of finite full subcategories. Then we show that stably equivalent ring functors have the same $T H H$. Some of these facts we have already used. The matrix ring functor was defined in 1.2.6 and we display the presence of a "trace" map in order to show Morita equivalence. Lastly we will show how THH may be calculated as Bökstedt's topological Hochschild homology of an FSP. Throughout this section we will be using ring functors without units as a computational tool, and the reader is referred to the end of the previous section for notation pertaining to presimplicial objects. In particular, remember that prehomotopic maps between simplicial sets give rise to homotopic maps on realization, and that specially homotopic maps between cyclic sets give rise to $C_{r}$ equivariant homotopy equivalences on realization for all $r$. Even though the procedure chosen is perhaps a bit non-standard, we hope that the reader will appreciate the fact that so many of the following questions can be taken care of by
repeating the same few tricks over and over again. This approach will, among other things, allow us to give a direct proof of Morita equivalence without reference to the standard bisimplicial space mapping by weak equivalences to both $T H H$ of the ring functor and the matrices.
1.6.1. Behavior of $T H H$ under natural isomorphisms and equivalences. We start out by noting that $T H H$ is insensitive to natural isomorphisms of functors. We say that two ring functors on $\mathscr{C}$, say $A_{0}$ and $A_{1}$, are isomorphic if there are natural isomorphisms of functors $A_{0} \rightarrow A_{1}$ and $A_{1} \rightarrow A_{0}$ compatible with the multiplicate structure (and unit if there is one). Likewise for (bi) modules. Up to natural isomorphism THH obviously does not see the difference. As a particular example consider the following:

If $\phi: \mathscr{D} \rightarrow \mathscr{C}$ in any functor and if $A$ is a ring functor on $\mathscr{C}$ and $P$ an $A$ bimodule we let $\phi^{*} A$ (resp. $\phi^{*} P$ ) denote the composite of $A$ (resp. $P$ ) with the functor $s_{*} \mathscr{E} n s^{\mathscr{C}^{\circ} \times ष} \rightarrow$ $s_{*} \mathscr{E} n s^{\mathscr{D}^{\circ} \times \mathscr{D}}$ induced by $\phi$. This is a ring functor on $\mathscr{D}$ (with unit if $A$ had one) and $\phi^{*} P$ is a (unital) $\phi^{*} A$ bimodule. We have a map $\bar{\phi}: T H H\left(\phi^{*} A, \phi^{*} P\right) \rightarrow T H H(A, P)$ given by sending the $\left(d_{0}, \ldots, d_{p}\right) \in \mathscr{D}^{p+1}$ summand of

$$
\bigvee_{\left(d_{0}, \ldots, d_{p}\right) \in \mathscr{Q}^{p+1}} P^{x_{0}}\left(\phi\left(d_{0}\right), \phi\left(d_{p}\right)\right) \wedge A^{x_{1}}\left(\phi\left(d_{1}\right), \phi\left(d_{0}\right)\right) \wedge \cdots \wedge A^{x_{p}}\left(\phi\left(d_{p}\right), \phi\left(d_{p-1}\right)\right)
$$

onto the $\left(\phi\left(d_{0}\right), \ldots, \phi\left(d_{p}\right)\right) \in \mathscr{C}^{p+1}$ summand of

$$
\bigvee_{\left(c_{0}, \ldots, c_{p}\right) \in 母^{p+1}} P^{x_{0}}\left(c_{0}, c_{p}\right) \wedge A^{x_{1}}\left(c_{1}, c_{0}\right) \wedge \cdots \wedge A^{x_{p}}\left(c_{p}, c_{p-1}\right)
$$

1.6.2. Lemma. Let $A$ be a ring functor on $\mathscr{C}$ and $P$ an $A$ bimodule. If $\phi_{0}$ and $\phi_{1}$ are naturally isomorphic functors from $\mathscr{D}$ to $\mathscr{C}$ then $T H H\left(\phi_{0}^{*} A, \phi_{0}^{*} P\right)$ and $T H H\left(\phi_{1}^{*} A, \phi_{1}^{*} P\right)$ are isomorphic.

Proof. Let $\eta: \phi_{0} \rightarrow \phi_{1}$ be the natural isomorphism. Then

$$
A^{X}\left(\eta_{a}^{-1}, \eta_{b}\right): A^{X}\left(\phi_{0}(a), \phi_{0}(b)\right) \rightarrow A^{X}\left(\phi_{1}(a), \phi_{1}(b)\right)
$$

(resp. $P^{X}\left(\eta_{a}^{-1}, \eta_{b}\right)$ ) induces a natural isomorphism between the functors $\phi_{0}^{*} A$ and $\phi_{1}^{*} A$ (resp. $\phi_{0}^{*} P$ and $\phi_{1}^{*} P$ ) compatible with the multiplication (and unit if $A$ has unit and $P$ is unital).
1.6.3. Corollary. If $P=A$ in the above lemma we have an isomorphism of cyclic objects.

However, one should be aware of the following subtle point. Although $T H H\left(\phi_{0}^{*} A, \phi_{0}^{*} P\right)$ and $T H H\left(\phi_{1}^{*} A, \phi_{1}^{*} P\right)$ are isomorphic, say by $\left(\eta^{-1}, \eta\right)^{*}$, the two maps

$$
T H H\left(\phi_{0}^{*} A, \phi_{0}^{*} P\right) \xrightarrow{\phi_{0}} T H H(A, P)
$$

and

$$
T H H\left(\phi_{0}^{*} A, \phi_{0}^{*} P\right) \xrightarrow{\left(\eta^{-1} \cdot \eta\right)^{*}} T H H\left(\phi_{1}^{*} A, \phi_{1}^{*} P\right) \xrightarrow{\bar{\phi}_{1}} T H H(A, P)
$$

are not in general equal. However we have that
1.6.4. Lemma. The maps $\bar{\phi}_{0}$ and $\bar{\phi}_{1}^{\circ}\left(\eta^{-1}, \eta\right)^{*}$ are prehomotopic.

Proof. We will define a map

$$
I I: T I I I I\left(\phi_{0}^{*} A, \phi_{0}^{*} P\right) \times N^{c y}(\mathscr{I}) \rightarrow T I I H(A, P)
$$

which upon restriction to $\Delta(1) \subseteq N^{c y}(\mathscr{J})$ is a prehomotopy from $\bar{\phi}_{0}$ to $\bar{\phi}_{1}{ }^{\circ}\left(\eta^{-1}, \eta\right)^{*}$. Let $\alpha=\left(i_{p} \leftarrow i_{0} \leftarrow i_{1} \leftarrow \cdots \leftarrow i_{p-1} \leftarrow i_{p}\right) \in N_{p}^{\mathrm{cy}}(\mathscr{I})$. Then $H_{\alpha}: T H H_{p}\left(\phi_{0}^{*} A, \phi_{0}^{*} P\right) \rightarrow$ $T H H_{p}(A, P)$ is given by (notation will be explained below)

$$
\xrightarrow[x \in I^{p+1}]{\text { holim }} \Omega_{\phi_{i_{0}}, \ldots, \phi_{i_{p}}}^{\sqcup_{x}} P^{x_{0}}\left(\eta_{d_{0}}^{-i_{0}}, \eta_{d_{p}}^{i_{p}}\right) \wedge A^{x_{1}}\left(\eta_{d_{1}}^{-i_{1}}, \eta_{d_{0}}^{i_{0} o}\right) \wedge \cdots \wedge A^{x_{p}}\left(\eta_{d_{p}}^{-i_{r}}, \eta_{d_{p-1}}^{i_{p-1}}\right) .
$$

Here $\bigvee_{\phi_{i_{0}}, \ldots, \phi_{i_{p}}}$ signifies that, say the $\left(d_{0}, \ldots, d_{p}\right) \in \mathscr{D}^{p+1}$ summand, is sent onto the $\left(\phi_{i_{0}}\left(d_{0}\right), \ldots, \phi_{i_{p}}\left(d_{p}\right)\right) \in \mathscr{C}^{p+1}$ summand. $\eta_{d_{k}}^{ \pm i_{k}}$ simply means $\eta_{d_{k}}^{-1}: \phi_{1}\left(d_{k}\right) \rightarrow \phi_{0}\left(d_{k}\right), i d_{\phi_{0}\left(d_{k}\right)}$ or $\eta_{d_{k}}: \phi_{0}\left(d_{k}\right) \leftarrow \phi_{1}\left(d_{k}\right)$ according to the value of $\pm i_{k}$. This assembles to give the desired prehomotopy.
1.6.5. Corollary. If $P=A$ in the lemma above, the prehomotopy of the proof yields that $\bar{\phi}_{0}$ and $\bar{\phi}_{1} \circ\left(\eta^{-1}, \eta\right)^{*}$ are specially homotopic.
1.6.6. Proposition. Let $\phi: \mathscr{D} \rightarrow \mathscr{C}$ be an equivalence of categories, $A$ a ring functor on $\mathscr{C}$ and $P$ an $A$ bimodule. Then

$$
\bar{\phi}: T H H\left(\phi^{*} A, \phi^{*} P\right) \rightarrow T H H(A, P)
$$

is a prehomotopy equivalence. If $P=A$ this is a special homotopy equivalence, and so if $A$ is unital this is a C-equivalence.

Proof. Let $\psi: \mathscr{C} \rightarrow \mathscr{D}$ be an inverse. Then

$$
T H H(A, P) \cong T H H\left(\psi^{*} \phi^{*} A, \psi^{*} \phi^{*} P\right) \xrightarrow{\underline{\phi} \Psi} T H H(A, P)
$$

and

$$
T H H\left(\phi^{*} A, \phi^{*} P\right) \xrightarrow{\bar{\phi}} T H H(A, P) \cong T H H\left(\psi^{*} \phi^{*} A, \psi^{*} \phi^{*} P\right) \xrightarrow{\Psi} T H H\left(\phi^{*} A, \phi^{*} P\right)
$$

are both prehomotopic (specially homotopic if $P=A$ ) to the identity.

In particular, this means that we can define $T H H$ uniquely up to special homotopy for all categories equivalent to small ones.
1.6.7. Example. Let $\phi: \mathscr{D} \rightarrow \mathscr{C}$ be an equivalence, and consider the ring functors on $\mathscr{C}$ of the type of Examples 1.2.2 and 1.2.3. More precisely, let $M$ be a subcategory of the category of sets containing the image of the morphism set functor $\mathscr{C}(-,-): \mathscr{C}^{\circ} \times \mathscr{C} \rightarrow M \subseteq \mathscr{E} n s$. Assume our ring functor factors as

$$
f s_{*} \mathscr{E} n s \xrightarrow{F} s_{*} \mathscr{E} n S^{M} \xrightarrow{\mathscr{\varphi}(-,-)^{*}} s_{*} \mathscr{E} n S^{\mathscr{E} \times \mathscr{C}} .
$$

By the proposition above get that $\operatorname{THH}\left(\phi^{*}\left(\mathscr{C}(-,-)^{*}\right) \circ F\right)$ and $T H H\left(\left(\mathscr{C}(-,-)^{*}\right)^{\circ} F\right)$ are specially homotopic. On the other hand, as $\phi$ is an equivalence of categories it induces a natural isomorphism between $\left(\mathscr{D}(-,-)^{*}\right) \circ F$ and $\phi^{*}\left(\mathscr{C}(-,-)^{*}\right) \circ F$, and so

$$
T H H\left(\left(\mathscr{D}(-,-)^{*}\right) \circ F\right) \cong T H H\left(\phi^{*}\left(\mathscr{C}(-,-)^{*}\right) \circ F\right) \xrightarrow{\cong} T H H\left(\left(\mathscr{C}(-,-)^{*}\right) \circ F\right)
$$

is a special homotopy equivalence.
This will be important in Chapter 2 where we consider THH as a functor from exact categories via the linear ring functor of Example 1.2.3, and so we get that $T H H$ sends natural equivalences to $C$-equivalences.
1.6.8. Topological Hochschild homology commutes with direct limits. Let $\mathscr{C}$ be a small category and let $J$ be a directed set of subcategories of $\mathscr{C}$ with the property that for any object $c \in \mathscr{C}$ there is a $j \in J$ such that $c \in j$. We then say that $J$ is a saturated directed set in $\mathscr{C}$. By abuse of notation we will use the same letter for an element in $J$ and its inclusion into $\mathscr{C}$.
1.6.9. Lemma. Let $J$ be a saturated directed set in $\mathscr{C}, A$ a ring functor on $\mathscr{C}$ and $P$ an A bimodule. Then the map

$$
\underset{j \in J}{\lim _{j \in J}} T H H\left(j^{*} A, j^{*} P\right) \rightarrow T H H(A, P)
$$

is an isomorphism.
Proof. The colimit commutes with the homotopy colimit, so for every $x \in I^{p+1}$ we consider the map

A map from $\left|S^{山 x}\right|$ to $|V(A, P)(x)|$ has compact image, and hence must stay within a finite number of summands. As $J$ is saturated and directed, there is a $j \in J$ such that the image is contained in summands indexed by elements in $j$. This gives the desired result.

If $j: \mathscr{D} \subseteq \mathscr{C}$ is any subcategory we will often write $\left.A\right|_{\mathscr{D}}$ (resp. $\left.\left.P\right|_{\mathscr{D}}\right)$ for $j^{*} A$ (resp. $j^{*} P$ ), the restriction to $\mathscr{D}$. Taking $J$ to be a set of finite subsets (i.e. discrete subcategories) of $o b \mathscr{C}$ such that any object in $\mathscr{C}$ is eventually contained in some $j \in J$, the lemma above tells us that $T H H(A, P)$ only depends on the values on finite discrete subcategories $\operatorname{THH}\left(\left.A\right|_{j}, P_{j}\right)$.
1.6.10. Stably equivalent ring functors. We have already used the following. Let $\phi: A_{1} \rightarrow A_{2}$ be a stable equivalence (see 1.1.1 for definition) of ring functors on $\mathscr{C}$, and $P_{1} \rightarrow P_{2}$ a stable equivalence of bimodules (meaning that $P_{1}$ is an $A_{1}$ bimodule and
$P_{2}$ is an $A_{2}$ bimodule（and hence an $A_{1}$ bimodule via $\phi$ ）and that there is a map $P_{1} \rightarrow P_{2}$ compatible with the bimodule structures over $A_{1}$ such that $\underline{\underline{P}}_{1} \rightarrow \underline{\underline{P}}_{2}$ is a stable equivalence．）Then we have

1．6．11．Lemma（ $T H H$ is invariant under stable equivalence）．The stable equivalence $\left(A_{1} \rightarrow A_{2}, P_{1} \rightarrow P_{2}\right)$ induces a weak equivalence $T H H\left(A_{1}, P_{1}\right) \xrightarrow{\simeq} T H H\left(A_{2}, P_{2}\right)$ ．If $P_{1}=A_{1}$ and $P_{2}=A_{2}$ this is a $C$－equivalence．

Proof．We immediately get that as the cardinality of $x \in I^{p+1}$ gets bigger，the connect－ ivity of

$$
\Omega^{\llcorner x} V\left(A_{1}, P_{1}\right)(x) \rightarrow \Omega^{\llcorner x} V\left(A_{2}, P_{2}\right)(x)
$$

grows to infinity．Hence $T H H_{p}\left(A_{1}, P_{1}\right) \rightarrow T H H_{p}\left(A_{2}, P_{2}\right)$ is a weak equivalence for all $p$ ．The first statement then follows by Lemma 1．5．10．

In the case where the ring functors themselves serve as bimodules，we will show that $\operatorname{sd_{r}}\left(T H H\left(A_{1}\right)\right)_{q}^{C_{r}} \rightarrow s d_{r}\left(T H H\left(A_{2}\right)\right)_{q}^{C_{r}}$ is a weak equivalence for all $q$ and the result follows as above．

$$
\begin{aligned}
& \left(s d_{r} T H H\left(A_{k}\right)^{C_{r}}\right)_{q-1}=\left(T H H_{r q-1}\left(A_{k}\right)\right)^{C r} \\
& =\left(\underset{x \in I^{r^{G}}}{\operatorname{holim}} \Omega^{\Delta x} V\left(A_{k}\right)(x)\right)^{c_{r}}=\underset{x \in I^{q}}{\operatorname{holim}}\left(\Omega^{\nu\left(x^{\nu r}\right)} V\left(A_{k}\right)\left(x^{\nu r}\right)\right)^{c}
\end{aligned}
$$

where $\boldsymbol{x}^{\sqcup r} \in I^{q r}$ is the image of the $r$－fold diagonal of $\boldsymbol{x} \in I^{q}$（which are the only fixed points under the $C_{r}$ action on $I^{r q}$ ）．Now，for any $C_{s} \subseteq C_{r}$ ，letting $t=r / s$

$$
\begin{aligned}
& V\left(A_{k}\right)\left(x^{\sqcup r}\right)^{C_{s}}= \bigvee_{\left(c_{1}, \ldots, c_{-t_{q}}\right) \in \mathscr{母}^{-t_{q}}} \\
&\left(\left(A_{k}^{x_{1}}\left(c_{1}, c_{-t_{q}}\right) \wedge \cdots \wedge A_{k}^{x_{q}}\left(c_{q}, c_{q-1}\right)\right.\right. \\
& \wedge\left(A_{k}^{x_{1}}\left(c_{q+1}, c_{q}\right) \wedge \cdots \wedge A_{k}^{x_{q}}\left(c_{2 q}, c_{2 q-1}\right) \wedge \cdots\right. \\
&\left.\wedge\left(A_{k}^{x_{1}}\left(c_{(t-1) q+1}, c_{(t-1) q}\right) \wedge \cdots \wedge A_{k}^{x_{q}}\left(c_{-t_{q}}, c_{t q-1}\right)\right)^{\wedge_{s}}\right)^{c_{s}} \\
& \cong V\left(A_{k}\right)\left(x^{山 t}\right) .
\end{aligned}
$$

By Lemma 1.6 .9 we may assume that $\mathscr{C}$ is finite，so to have some concrete numbers to work with，say that $A_{1}^{n}(a, b) \rightarrow A_{2}^{n}(a, b)$ is $2 n-c$ connected for some constant $c$（de－ pending on neither $a, b$ nor $n$ ）．Then $V\left(A_{1}\right)\left(x^{\omega_{r}}\right)^{c_{s}} \rightarrow V\left(A_{2}\right)\left(x^{山_{r}}\right)^{c_{s}}$ is $t \sum\left(2 x_{i}-c\right)$－ connected，and hence by Lemma 3.11 of［3］
is $t \sum\left(x_{i}-c\right)$ connected．In particular，if $r=s$ this is $\sum\left(x_{i}-c\right)$－connected and by Lemma 1．5．10 we are done．

1．6．12．Preservation of products．In this subsection assume we are given two small categories $\mathscr{C}_{1}$ and $\mathscr{C}_{2}$ and assume that $A_{1} \in \mathscr{F} \mathscr{C}_{1}$ and $A_{2} \in \mathscr{F} \mathscr{C}_{2}$ and that $P_{1}$ and $P_{2}$ are $A_{1}$ and $A_{2}$ bimodules．For the time being we will not assume the presence of
a unit. Recall the definitions of $A_{1} \times A_{2}, A_{1} \vee A_{2}$ and $A_{1} \amalg A_{2}$ (see 1.2.5 and 1.2.10). Similarly we can define the bimodules $P_{1} \times P_{2}, P_{1} \vee P_{2}$ and $P_{1} 山 P_{2}$. In the unital case, the first and last bimodules will be unital as well, but the wedge will not.
Let $\boldsymbol{x}=\left(x_{0}, \ldots, x_{p}\right) \in I^{p+1}$. Note that $V\left(A_{1} \amalg A_{2}, P_{1} \amalg P_{2}\right)(x)=V\left(A_{1}, P_{1}\right)(x) V$ $V\left(A_{2}, P_{2}\right)(\boldsymbol{x})$ (the only summands surviving in $V\left(A_{1} \amalg A_{2}, P_{1} \amalg P_{2}\right)(\boldsymbol{x})$ are the ones where all $c_{0}, \ldots, c_{p}$ lie in the same component of $\left.\mathscr{C}_{1} \amalg \mathscr{C}_{2}\right)$. Hence the inclusion $V\left(A_{1}, P_{1}\right)(x) \vee V\left(A_{2}, P_{2}\right)(x) \subseteq V\left(A_{1}, P_{1}\right)(x) \times V\left(A_{2}, P_{2}\right)(x)$ induces a map $j: T H H$ $\left(A_{1} \amalg A_{2}, P_{1} \amalg P_{2}\right) \rightarrow \operatorname{THH}\left(A_{1}, P_{1}\right) \times \operatorname{THH}\left(A_{2}, P_{2}\right)$.
1.6.13. Lemma. Let $A_{1} \in \mathscr{F} \mathscr{C}_{1}$ and $A_{2} \in \mathscr{F} \mathscr{C}_{2}$ and let $P_{1}$ and $P_{2}$ be $A_{1}$ and $A_{2}$ bimodules. Then

$$
j: T H H\left(A_{1} \amalg A_{2}, P_{1} \amalg P_{2}\right) \rightarrow T H H\left(A_{1}, P_{1}\right) \times T H H\left(A_{2}, P_{2}\right)
$$

is a weak equivalence. If $P_{1}=A_{1}$ and $P_{2}=A_{2}$ it induces a $C$-equivalence, and hence in the unital case $|j|$ is a $C_{r}$ equivariant homotopy equivalence for all $r$.

Proof. Blakers-Massey guarantees that

$$
V\left(A_{1}, P_{1}\right)(x) \vee V\left(A_{2}, P_{2}\right)(x) \subseteq V\left(A_{1}, P_{1}\right)(x) \times V\left(A_{2}, P_{2}\right)(x)
$$

is $2(\sqcup x)-2 p-3$ connected, and so $j$ is a weak equivalence.
For the $P_{k}=A_{k}$ case consider for all $q>0$ and subgroups $C_{r} \subseteq C_{q}$ the map

$$
j_{q-1}^{c_{r}}: T H H_{q}\left(A_{1} \amalg A_{2}\right)^{C_{r}} \rightarrow\left(T H H_{q-1}\left(A_{1}\right) \times T H H_{q-1}\left(A_{2}\right)\right)^{C_{r}} .
$$

Letting $p=q / r$ this is just the map:

$$
\begin{aligned}
& \xrightarrow[x \in I^{P}]{\text { holim }}\left(\Omega^{U^{L r}} V\left(A_{1}\right)\left(x^{U r}\right) \vee V\left(A_{2}\right)\left(x^{U r}\right)\right)^{C_{r}} \\
& \rightarrow \underset{x \in I^{P}}{\text { holim }}\left(\Omega^{\left\llcorner^{\Delta^{L}}\right.} V\left(A_{1}\right)\left(x^{\llcorner r}\right) \times V\left(A_{2}\right)\left(x^{\llcorner r}\right)\right)^{c r}
\end{aligned}
$$

Now,

$$
\left(V\left(A_{1}\right)\left(x^{\llcorner r}\right) \vee V\left(A_{2}\right)\left(x^{\llcorner r}\right)\right)^{c r} \cong V\left(A_{1}\right)(x) \vee V\left(A_{2}\right)(x)
$$

and likewise for the product, so

$$
\left(V\left(A_{1}\right)\left(x^{\llcorner r}\right) \vee V\left(A_{2}\right)\left(x^{\llcorner r}\right)\right)^{C r} \rightarrow\left(V\left(A_{1}\right)\left(x^{\Delta r}\right) \times V\left(A_{2}\right)\left(x^{\Delta r}\right)\right)^{c_{r}}
$$

is $2(\sqcup \boldsymbol{x})-2 p-3$ connected (and similarly for the subgroups of $C_{r}$ ), and so by [3, Lemma 3.11] the map $\left(\Omega^{\Delta x^{\llcorner r}} V\left(A_{1}\right)\left(x^{\Delta r}\right) \vee V\left(A_{2}\right)\left(x^{\Delta r}\right)\right)^{C r} \rightarrow\left(\Omega^{\Delta x^{\Delta r}} V\left(A_{1}\right)\left(x^{\Delta r}\right) \times\right.$ $\left.V\left(A_{2}\right)\left(x^{\llcorner r}\right)\right)^{c_{r}}$ is $\left(\sqcup_{x}\right)-2 p-3$ connected.

Hence it follows that $s d_{r} j_{p}^{C_{r}}=j_{q-1}^{C_{r}}$ is a weak equivalences for all $q>0$ and $C_{r} \subseteq C_{q}$. By Lemma 1.5.10 and the equivariant Whitehead lemma, the result follows.
1.6.14. Notation. From now we will need precise notation for handling the various indexations for $V(A, P)(x)$. If $E$ is some set, we let $\{E\}_{p}$ denote the set of functions
$Z /(p+1) Z \rightarrow E$. If $e \in\{E\}_{p}$ we write $e_{i}$ for $e(i)$. If we write $\{\mathscr{C}\}_{p}$ where $\mathscr{C}$ is some small category, we shall mean $\left\{o b^{\mathscr{C}}\right\}_{p}$. We shall choose indexation such that the face maps (and cyclic permutations if $P=A$ ) on $V(A, P)(x)$ corresponds to the cyclic actions on $\{E\}_{p}$. Thus for instance, using that smash is distributive over wedge up to natural isomorphism, we get that $V\left(A_{1} \vee A_{2}, P_{1} \vee P_{2}\right)(x)$ could be written out as

$$
\bigvee_{\left\{\left(c^{1}, c^{2}, r\right) \in \mathscr{母}^{1} \times \mathscr{母}^{2} \times\{1,2\}\right\}_{p}} P_{r_{0}}^{x_{0}}\left(c_{0}^{r_{0}}, c_{p}^{r_{0}}\right) \wedge A_{r_{1}}^{x_{1}}\left(c_{1}^{r_{1}}, c_{0}^{r_{1}}\right) \wedge \cdots \wedge A_{r_{p}}^{x_{p}}\left(c_{p}^{r_{p}}, c_{p-1}^{r_{p}}\right)
$$

The natural map $f_{x}: V\left(A_{1} \times A_{2}, P_{1} \times P_{2}\right)(\boldsymbol{x}) \rightarrow V\left(A_{1}, P_{1}\right)(x) \times V\left(A_{2}, P_{2}\right)(x)$ restricts to a projection $g_{x}: V\left(A_{1} \vee A_{2}, P_{1} \vee P_{2}\right)(x) \rightarrow V\left(A_{1}, P_{1}\right)(x) \vee V\left(A_{2}, P_{2}\right)(x)$ which we may give a section as follows. In the notation above $g_{x}$ is given by sending the $\left(c^{1}, c^{2}, r\right)$ th summand to the basepoint if $r$ is not constant, and onto the $r$ th summand if $r$ is either $\equiv 1$ or $\equiv 2$. Choose some fixed but arbitrary object $(a, b) \in \mathscr{C}_{1} \times \mathscr{C}_{2}$ and define a section $i_{x}: V\left(A_{1}, P_{1}\right)(x) \vee V\left(A_{2}, P_{2}\right)(x) \rightarrow V\left(A_{1} \vee A_{2}, P_{1} \vee P_{2}\right)(x)$ by sending the $\left(c_{0}^{1}, \ldots, c_{p}^{1}\right) \in \mathscr{C}_{1}^{p+1} \subseteq\left(\mathscr{C}_{1} \amalg \mathscr{C}_{2}\right)^{p+1}$ onto the $\left(c^{1}, b, 1\right)$ summand, and likewise for the other summands. This defines a simplicial map $i: \operatorname{THH}\left(A_{1} \amalg A_{2}, P_{1} \amalg P_{2}\right)(x) \rightarrow$ $T H H\left(A_{1} \vee A_{2}, P_{1} \vee P_{2}\right)$. We use this to prove:
1.6.15. Proposition (Preservation of products). Let $A_{1} \in \mathscr{F} \mathscr{C}_{1}^{u}$ and $A_{2} \in \mathscr{F} \mathscr{C}_{2}^{u}$ and let $P_{1}$ and $P_{2}$ be unital bimodules. Then

$$
T H H\left(A_{1} \times A_{2}, P_{1} \times P_{2}\right) \xrightarrow{f} T H H\left(A_{1}, P_{1}\right) \times T H H\left(A_{2}, P_{2}\right)
$$

is a weak equivalence. If $P_{1}=A_{1}$ and $P_{2}=A_{2}$, then $f$ is a $C$-equivalence, and so $|f|$ is a $C_{r}$ equivariant homotopy equivalence for every $r>0$.

Proof. Let $X=T H H\left(A_{1} \times A_{2}, P_{1} \times P_{2}\right), \quad Y=T H H\left(A_{1}, P_{1}\right) \times \operatorname{THH}\left(A_{2}, P_{2}\right), \quad V=$ $T H H\left(A_{1} \vee A_{2}, P_{1} \vee P_{2}\right)$ and $W=T H H\left(A_{1} \amalg A_{2}, P_{1} \amalg P_{2}\right)$, and consider the diagram


We want to show that all maps are weak equivalences, and that if $P_{k}=A_{k}$ we are in the situation of Lemma 1.5.12.

By Blakers-Massey the product and wedge ring functors and bimodules are stably homotopic so by Lemma 1.6 .11 the map $V \rightarrow X$ is a degreewise weak equivalence, and if $P_{k}=A_{k}$ (for $k=1$ and 2) a degreewise weak equivalence on $s d_{r}(-)^{C_{r}}$.

As to the lower horizontal map, we have defined a section $i$ of $g$, so $g \circ i=i d_{W}$ and we define a map $H: V \times N^{c y}(\mathscr{I}) \rightarrow V$ which upon restriction to $\Delta(1) \subseteq N^{c y}(\mathscr{I})$ is a prehomotopy from the identity to $i \circ g$. If $\alpha=\left(i_{p} \leftarrow i_{0} \leftarrow i_{1} \leftarrow \cdots i_{p-1} \leftarrow i_{p}\right) \in N_{p}^{c y}(\mathscr{I})$
let $\phi_{a}$ be the self-map of the indexing set $\left\{\left(\mathscr{C}_{1} \times \mathscr{C}_{2}\right) \times\{1,2\}\right\}_{p}$ given by

$$
\phi_{\alpha}\left(c^{1}, c^{2}, r\right)(k)= \begin{cases}\left(\mathrm{c}_{k}^{1}, b, 1\right) & \text { if } r_{k}=1 \text { and } i_{k}=0 \\ \left(a, c_{k}^{2}, 2\right) & \text { if } r_{k}=2 \text { and } i_{k}=0 \\ \left(c_{k}^{1}, c_{k}^{2}, r_{k}\right) & \text { if } i_{k}=1\end{cases}
$$

Then we define $H_{\alpha, x}: V\left(A_{1} \vee A_{2}, P_{1} \vee P_{2}\right)(x) \rightarrow V\left(A_{1} \vee A_{2}, P_{1} \vee P_{2}\right)(x)$ by sending $y$ in the $\left(c^{1}, c^{2}, r\right)$ summand to

$$
H_{\alpha, x}(y)= \begin{cases}y & \text { in the } \phi_{\alpha}\left(c^{1}, c^{2}, r\right) \text { summand if } i_{k}=0 \Rightarrow r_{k}=r_{k, 1}, \\ * & \text { otherwise }\end{cases}
$$

Applying $\Omega^{\sqcup x}$ and going to the limit this defines, as $p$ varies, a prehomotopy when restricting to $\Delta(1) \subseteq N^{\mathrm{cy}}(\mathscr{I})$ and if $P_{k}=A_{k}$ a special homotopy (to see this notice that $d_{j} y=*$ if $r_{j} \neq r_{j+1}$ ). Hence $V \rightarrow W$ is a prehomotopy equivalence and if $P_{k}=A_{k}$ a special homotopy equivalence. Together with the above lemma we have now proven that all maps are weak equivalences, and collecting everything we have proven for the $P_{k}=A_{k}$ part we see that we are in the situation of Lemma 1.5.12 and the result follows.

Comment. After such a lengthy proof it is worthwhile to extract the ideology used. The following should not be considered as a part of this section (or even this paper), but is meant as an aid for the interested reader.

Let $P=A$ for simplicity. As advertised many times, the crucial step in this approach was that we exchanged the original ring functors with more accessible ones without unit. The accessibility lies in the fact that, in the notation above, we were able to give a "simple" description of a splitting of the map $V \rightarrow W$, and furthermore, we could give an explicit homotopy

$$
V \times N^{\mathrm{cy}}(\mathscr{I}) \rightarrow V
$$

The latter was built on the following idea which we shall have occasion to use many times. Think of an element $\alpha=\left(i_{p} \leftarrow i_{0} \leftarrow \cdots \leftarrow i_{p}\right) \in N_{p}^{\text {cy }}(\mathscr{I})$ as a string of actions on $A^{x_{0}}\left(c_{0}, c_{p}\right) \wedge \cdots \wedge A^{x_{p}}\left(c_{p}, c_{p-1}\right)$ testing compatibility between two pairs. By this we mean the following: "if $i_{k}$ equals zero, test whether $A^{x_{k+1}}\left(c_{k+1}, c_{k}\right)$ and $A^{x_{k}}\left(c_{k}, c_{k-1}\right)$ always multiply trivially. If it does, send everything to the basepoint, but otherwise leave it as it is." This is a presimplicial (even precyclic) gadget, for taking any other face map but the $k$ th, we still have the same potential question (moved appropriately); and if we use the $k$ th face map, the knowledge of whether the question was asked disappears, but this becomes irrelevant for if it would have been answered affirmatively we are sent to the basepoint in either case.

Thus if the difference between two self-maps on $V(A)(x)$ can be expressed by some criterion of "non-compatibility" implying trivial multiplication, we are able to bridge them by a homotopy, as examplified with $A=A_{1} \vee A_{2}$ in the proof. Here this criterion was that the index did not stay within one of the categories (and hence forcing some
face map to be trivial). Of course, there is the additional problem with putting the final answer in the right summand, but this should be regarded as technical. For an example not containing this extra difficulty see below. However, there is one point related to this that we should be aware of. The exchange of wedges for products allowed us to use the natural distributivity of smash over wedges, making it possible to write out $V(A)(x)$ as a wedge over an indexing set. It is important that we make this indexation compatible with the precyclic actions. More precisely: the index sets are maps from $Z /(p+1) Z$ and so cyclic sets themselves, and the face maps and cyclic operators of $V(A)(x)$ should send summands onto each other in accordance with the operations on the indexation. We see that this is the case with the indexation $\left\{\mathscr{C}^{1} \times \mathscr{C}^{2} \times\{1,2\}\right\}_{p}$ of $V\left(A_{1} \vee A_{2}\right)(x)$. If we have an element $y$ in the $\left(c^{1}, c^{2}, r\right)$ summand then $d_{j} y$ naturally lies in the $d_{j}\left(c^{1}, c^{2}, r\right)$ summand (note that if $r_{j} \neq r_{j+1}$ it is sent to the basepoint).

We also have to check that the reindexation $\phi$ is a precyclic map (in the sense that $d_{j} \phi_{\alpha}=\phi_{d \alpha} d_{j}$ and likewise for the cyclic operator). Now it is easy to see that

$$
d_{j} H_{\alpha, x}(y)= \begin{cases}d_{j} y & \text { in the } d_{j} \phi_{\alpha}\left(c^{1}, c^{2}, r\right) \text { summand if for } 0 \leq k \leq p \\ & i_{k}=0 \Rightarrow r_{k}=r_{k+1} \\ * & \text { otherwise }\end{cases}
$$

and

$$
H_{d_{j} \alpha, d_{j} x}\left(d_{j} y\right)= \begin{cases}d_{j} y & \text { in the } \phi_{d_{j} x} d_{j}\left(c^{1}, c^{2}, r\right) \text { summand if for } \\ 0 \leq k \neq j \leq p, i_{k}=0 \Rightarrow r_{k}=r_{k+1}, \\ * & \text { otherwise }\end{cases}
$$

are equal:

| $i_{j}$ | $r_{j}=r_{j+1}$ | $H_{d j a, d j x}\left(d_{j} y\right)=d_{j} H_{\alpha, x}(y)$ |  |
| :--- | :--- | :--- | :--- |
| 0 | Yes | Yes | (Automatic) |
| 0 | No | Yes | (both $=*$ ) |
| 1 | Yes | Yes | (No $r_{j}=r_{j+1}$ condition |
| 1 | No | Yes | on $H_{\alpha, x}(y)$ either) |

All this testing is implicit in the text proper, and will not be carried through from now on.
It is perhaps noteworthy that a similar, but somewhat easier proof yields that if we had used the internal product of ring functors we would have had the same statement. More precisely: if $\mathscr{C}=\mathscr{C}_{1}=\mathscr{C}_{2}$ in the above situation, we define the internal product of $A_{1}$ and $A_{2}$ to be $\operatorname{diag}^{*}\left(A_{1} \times A_{2}\right)$ where $\operatorname{diag}: \mathscr{C} \rightarrow \mathscr{C} \times \mathscr{C}$ is the diagonal inclusion. There is a map $f^{\circ} \overline{\mathrm{diag}}: T H H\left(\operatorname{diag}^{*}\left(A_{1} \times A_{2}\right)\right.$, $\left.\operatorname{diag}^{*}\left(P_{1} \times P_{2}\right)\right) \rightarrow T H H\left(A_{1}, P_{1}\right) \times$ $T H H\left(A_{2}, P_{2}\right)$, and laziness dictates that we call it simply $f$ (for the definition of $\bar{\phi}$ see 1.6.1). We still get that

$$
T H H\left(\operatorname{diag}^{*}\left(A_{1} \vee A_{2}\right), \operatorname{diag}^{*}\left(P_{1} \vee P_{2}\right)\right) \rightarrow T H H\left(\operatorname{diag}^{*}\left(A_{1} \times A_{2}\right), \operatorname{diag}^{*}\left(P_{1} \times P_{2}\right)\right)
$$

is induced by stable equivalences, and what one must show is that

$$
T H H\left(\operatorname{diag}^{*}\left(A_{1} \vee A_{2}\right), \operatorname{diag}^{*}\left(P_{1} \vee P_{2}\right)\right) \quad \text { and } \quad T H H\left(\left(A_{1} \amalg A_{2}\right),\left(P_{1} \amalg P_{2}\right)\right)
$$

are prehomotopy equivalent (specially homotopy equivalent if $P_{k}=A_{k}$ ).
So writing this out, we get that $V\left(\operatorname{diag}^{*}\left(A_{1} \times A_{2}\right)\right.$, $\left.\operatorname{diag}^{*}\left(P_{1} \times P_{2}\right)\right)(x)$ equals

$$
\bigvee_{(c, r) \in\{\& \times\{1,2\}\}_{p}} P_{r_{0}}^{x_{0}}\left(c_{0}, c_{n}\right) \wedge A_{r_{1}}^{x_{1}}\left(c_{1}, c_{0}\right) \wedge \cdots \wedge A_{r_{p}}^{x_{p}}\left(c_{p}, c_{p-1}\right)
$$

Let $\quad i_{x}: V\left(A_{1} \amalg A_{2}, P_{1} \amalg P_{2}\right)=V\left(A_{1}, P_{1}\right)(x) \vee V\left(A_{2}, P_{2}\right)(x) \rightarrow V\left(\operatorname{diag}^{*}\left(A_{1} \vee A_{2}\right)\right.$, $\left.\operatorname{diag}^{*}\left(P_{1} \vee P_{2}\right)\right)(x)$ be the inclusion and $g_{x}: V\left(A_{1} \vee A_{2}, P_{1} \vee P_{2}\right)(x)=V\left(A_{1}, P_{1}\right)(x) \vee$ $V\left(A_{2}, P_{2}\right)(x)$ be the splitting given by sending all summands but the ones with constant $r$ 's to the basepoint. This induces a map

$$
g: T H H\left(\operatorname{diag}^{*}\left(A_{1} \vee A_{2}\right), \operatorname{diag}^{*}\left(P_{1} \vee P_{2}\right)\right) \rightarrow T H H\left(A_{1} \amalg A_{2}, P_{1} \amalg P_{2}\right)
$$

(which is simply $g \circ \overline{\text { diag }}$ in the in the notation of 1.6 .14 ) with section $i$
1.6.16. Lemma (Preservation of internal product). Let $A_{1}$ and $A_{2}$ be two ring functors on $\mathscr{C}$ with unit and let $P_{1}$ and $P_{2}$ be unital bimodules. Then

$$
T H H\left(\operatorname{diag}^{*}\left(A_{1} \times A_{2}\right), \operatorname{diag}^{*}\left(P_{1} \times P_{2}\right)\right) \xrightarrow{f} T H H\left(A_{1}, P_{1}\right) \times T H H\left(A_{2}, P_{2}\right)
$$

is a weak equivalence. If $P_{1}=A_{1}$ and $P_{2}=A_{2}$ then it is a $C$-equivalence.

Proof. The only thing remaining to demonstrate is that we have a map

$$
\begin{aligned}
H: & T H H\left(\operatorname{diag}^{*}\left(A_{1} \vee A_{2}\right), \operatorname{diag}^{*}\left(P_{1} \vee P_{2}\right)\right) \times N^{c y}(\mathscr{I}) \\
& \rightarrow T H H\left(\operatorname{diag}^{*}\left(A_{1} \vee A_{2}\right), \operatorname{diag}^{*}\left(P_{1} \vee P_{2}\right)\right)
\end{aligned}
$$

which upon restriction to $\Delta(1) \subseteq N^{\text {cy }}(\mathscr{I})$ is a prehomotopy from the identity to $i \circ g$ and which in the $P_{k}=A_{k}$ case is a special homotopy. If $\alpha=\left(i_{p} \leftarrow i_{0} \leftarrow\right.$ $\left.i_{1} \leftarrow \cdots i_{p-1} \leftarrow i_{p}\right) \in N_{p}^{\text {cy }}(\mathscr{I})$ and $y$ is in the $(c, r) \in\{\mathscr{C} \times\{1,2\}\}_{p}$ summand we define

$$
H_{\alpha, x}(y)= \begin{cases}y & \text { if } i_{k}=0 \Rightarrow r_{k}=r_{k+1} \\ * & \text { otherwise }\end{cases}
$$

Applying $\Omega^{山_{x}}$ and going to the limit, as $p$ varies this defines a prehomotopy and if $P_{k}=A_{k}$ a special homotopy.
1.6.17. Morita Equivalence. Let $A$ be a ring functor on $\mathscr{C}$ and $P$ an $A$ bimodule. Recall from Section 1.2 .6 the definition of $M_{n} A$ and $\left(M_{n} A\right)_{V}$. In the same fashion $M_{n} P$ (resp. $\left.\left(M_{n} P\right)_{V}\right)$ is the $M_{n} A$ (resp. $\left.\left(M_{n} A\right)_{V}\right)$ bimodule given by

$$
\begin{aligned}
& \left(M_{n} P\right)^{X}(C, D)=\prod_{r \in n} \bigvee_{s \in n} P^{X}\left(p r_{r} C, p r_{s} D\right) \\
& \left(\text { resp. }\left(M_{n} P\right)_{V}^{X}(C, D)=\bigvee_{(r, s) \in n^{2}} P^{X}\left(p r_{r} C, p r_{s} D\right)\right)
\end{aligned}
$$

with the obvious $M_{n} A$ (resp. $\left.\left(M_{n} A\right)_{V}\right)$ action. The inclusions $\left(M_{n} A\right)_{V} \rightarrow M_{n} A$ and $\left(M_{n} P\right)_{\vee} \rightarrow M_{n} P$ are stable equivalences, so for all practical purposes we may just as well work with the simpler wedge construction.

Choose some arbitrary but fixed element $a \in \mathscr{C}$. Let in: $\mathscr{C} \rightarrow \mathscr{C}^{n}$ be the inclusion into the first coordinate given by sending an arrow $f$ in $\mathscr{C}$ to $\left(f, i d_{a}, \ldots, i d_{a}\right)$ (it is not important that we use the identity on $a$, any idempotent would do, in particular if we have some sort of zero, we may use this instead). We have a map of ring functors on $\mathscr{C} \eta: A \rightarrow i n^{*}\left(M_{n} A\right)_{\vee}$ by the inclusion

$$
\begin{aligned}
& A^{X}(c, d) \rightarrow A^{X}(c, d) \vee \bigvee_{n-1} A^{X}(c, a) \vee \bigvee_{n-1} A^{x}(a, d) \vee \bigvee_{(n-1)^{2}} A^{x}(a, a) \\
& \quad=\left(M_{n} A\right)_{V}^{x}(i n(c), i n(d))
\end{aligned}
$$

$\eta$ has a splitting $\varepsilon: i^{*}\left(M_{n} A\right)_{\vee} \rightarrow A$ by the corresponding projection. Likewise for $P$. This induces

$$
i: T H H(A, P) \xrightarrow{\eta} T H H\left(i n^{*}\left(M_{n} A\right)_{\vee}, i n^{*}\left(M_{n} P\right)_{\vee}\right) \xrightarrow{i n} T H H\left(\left(M_{n} A\right)_{\vee},\left(M_{n} P\right)_{\vee}\right) .
$$

Likewise we have inclusions of $A$ into $i n^{*} M_{n} A$ and $P$ into $i n^{*} M_{n} P$, and these are unital if $A$ and $P$ are. These inclusions factor through $\varepsilon$ followed by the stable equivalence induced from including the wedges into the products. Let $j: T H H(A, P) \rightarrow$ $T H H\left(M_{n} A, M_{n} P\right)$ denote the induced map.

The advantage is $i$ is that there is an easy "trace" map back, which we will now define. First write out $V\left(\left(M_{n} A\right)_{V},\left(M_{n} P\right)_{V}\right)(x)$ :

$$
\bigvee_{(r, s, C) \in\left\{n^{2} \times 母^{m}\right\}_{p}} p^{x_{0}}\left(p r_{r 0} C_{0}, p r_{s_{p}} C_{p}\right) \wedge \cdots \wedge A^{x_{p}}\left(p r_{r_{p}} C_{p}, p r_{s_{p-1}} C_{p-1}\right)
$$

(see 1.6 .14 for notation). Thus the inclusion above is given by sending the $\left(c_{0}, \ldots, c_{p}\right)$ summand of $V(A, P)(x)$ onto the $(r, s, C)$ summand with $C_{k}=\left(c_{k}, a, \ldots, a\right)$ and $r=s=1$.

Then the trace map is defined by sending $y$ in the $(r, s, C)$ summand to

$$
\operatorname{Tr}_{x}(y)= \begin{cases}* & \text { if } r_{k} \neq s_{k} \text { for some } k, \\ y & \text { in the }\left\{k \mapsto p r_{r_{k}} C_{k}\right\} \text { summand if } r_{k}=s_{k} \text { for all } k\end{cases}
$$

This defines a map $\operatorname{Tr}: \operatorname{THH}\left(\left(M_{n} A\right)_{\mathrm{V}},\left(M_{n} P\right)_{\mathrm{V}}\right) \rightarrow \operatorname{THH}(A, P)$, with $\operatorname{Tr} \circ i=i d$.
1.6.18. Proposition (Morita equivalence). Let $A$ be a unital ring functor on $\mathscr{C}, P a$ unital $A$ bimodule and $n \in N$. The inclusions defined above yield weak equivalences $i: T H H(A, P) \rightarrow T H H\left(M_{n} A, M_{n} P\right)$. If $P=A$ this is a $C$-equivalence.

Proof. The framework of the proof is exactly as for the proof of preservation of product. Let $X=V=T H H(A, P), Y=T H H\left(M_{n} A, M_{n} P\right)$, and $W=T H H\left(\left(M_{n} A\right)_{\vee} \times\right.$, $\left.\left(M_{n} P\right)_{V}\right)$, and consider the diagram


We want to show that all maps are weak equivalences, and that if $P=A$ we are in the situation of Lemma 1.5.12.

Now, $\operatorname{Tr} \circ i=i d$, and we will define a presimplicial homotopy (special if $P=A$ ) from $i \circ \operatorname{Tr}$ to the identity. For each $\alpha=\left(i_{p} \leftarrow i_{0} \leftarrow \cdots \leftarrow i_{p-1} \leftarrow i_{p}\right) \in N_{p}^{\text {cy }}(\mathscr{I})$ we define a map $H_{x . x}: V\left(M_{n} A, M_{n} P\right)(x) \rightarrow V\left(M_{n} A, M_{n} P\right)(x)$ as follows. Let $\phi_{\alpha}$ be the selfmap of the indexing set $\left\{n^{2} \times \mathscr{C}\right\}_{p}=\mathscr{E} n s\left(Z /(p+1) Z, n^{2} \times \mathscr{C}\right)$ given by

$$
\phi_{a}(r, s, C)(k)= \begin{cases}\left(1,1, i n \circ p r_{r_{k}} C_{k}\right) & \text { if } i_{k}=0 \\ \left(r_{k}, s_{k}, C_{k}\right) & \text { if } i_{k}=1\end{cases}
$$

Now, if $y$ is in the $(r, s, C)$ summand

$$
\begin{aligned}
& H_{\alpha, x}(y) \\
& \quad= \begin{cases}y \text { in the } \phi_{\alpha}(r, s, C) \text { summand } & \text { if } i_{k}=0 \Rightarrow r_{k}=s_{k} \text { for all } 0 \leq k \leq p \\
* & \text { otherwise }\end{cases}
\end{aligned}
$$

This assembles to a presimplicial homotopy from $i \circ \operatorname{Tr}$ to the identity. Note that if $P=A$ this actually defines a special homotopy.
1.6.19. Upper triangular matrices. In the previous subsection we saw that the trace gave rise to an equivalence between the $T H H$ of a ring functor and its matrices. If we restrict our attention to upper triangular matrices we note that the only sequences giving non-zero contribution to the generalized trace are in fact the ones deriving from diagonal elements. This is so because once one is outside the diagonal, the next element in the generalized trace must be on the other side of the diagonal. This observation may make the the proposition below more plausible, and will be apparent in the proof itself.

Let $A \in \mathscr{F} \mathscr{C}$ and let $P$ be a bimodule. Recall from Section 1.2.7 the definition of $T_{n} A$ and $\left(T_{n} A\right)_{\mathrm{V}}$. In the same fashion $T_{n} P$ (resp. $\left.\left(T_{n} P\right)_{\vee}\right)$ is the $T_{n} A$ (resp. $\left.\left(T_{n} A\right)_{V}\right)$ bimodule given by

$$
\begin{aligned}
& \left(T_{n} P\right)^{X}(C, D)=\prod_{r \in \mathrm{n}} \bigvee_{s \leq r \in n} P^{X}\left(p r_{r} D, p r_{s} D\right) \\
& \left(\operatorname{resp} .\left(T_{n} P\right)_{V}^{X}(C, D)=\bigvee_{s \leq r \in n^{2}} P^{X}\left(p r_{r} C, p r_{s} D\right)\right)
\end{aligned}
$$

with the obvious $T_{n} A$ (resp. $\left.\left(T_{n} A\right)_{V}\right)$ action. The inclusions $\left(T_{n} A\right)_{\vee} \rightarrow T_{n} A$ and $\left(T_{n} P\right)_{\vee} \rightarrow T_{n} P$ are stable equivalences. Indeed in this setting have a diagram of ring functors on $\mathscr{C}^{n}$

where $i$ is given by the inclusion of the diagonal, and $p$ by collapsing of offdiagonal summands. This is possible as no off-diagonal elements may multiply to give a diagonal element. We define the same maps on the bimodules giving us split injections

$$
\begin{aligned}
& T H H\left(\prod_{k \in n} A, \prod_{k \in n} P\right) \xrightarrow{i} \operatorname{THH}\left(T_{n} A, T_{n} P\right) \\
& T H H\left(\bigvee_{k \in n} A, \bigvee_{k \in n} P\right) \xrightarrow{i} \operatorname{THH}\left(\left(T_{n} A\right)_{\mathrm{V},}\left(T_{n} P\right)_{V}\right) .
\end{aligned}
$$

Using the trace map again, we will show that the wedge models are specially homotopy equivalent. Note that even though both the inclusion and projection are defined if we use the product representations, the homotopy defined in the proof will not (be defined).
1.6.20. Proposition. Let $A$ be a ring functor with unit on $\mathscr{C}, P$ a unital $A$ bimodule, and $n \in N$. Then the inclusion $i: T H H\left(\prod_{k \in n} A, \prod_{k \in n} P\right) \rightarrow T H H\left(T_{n} A, T_{n} P\right)$ is split weak equivalence. If $P=A$ then this is a $C$-equivalence.

Proof. Now let $X=\operatorname{THH}\left(\prod_{n} A, \prod_{n} P\right), Y=\operatorname{THH}\left(T_{n} A, T_{n} P\right), V=T H H\left(V_{n} A, V_{n} P\right)$ and $W=T H H\left(\left(T_{n} A\right)_{\vee},\left(T_{n} P\right)_{\vee}\right)$, and consider the diagram


We want to show that all maps are weak equivalences, and that if $P=A$ we are in the situation of Lemma 1.5.12.

By Blakers-Massey the ring functors and bimodules defined by product or wedge are stably homotopy equivalent so by Lemma 1.6 .11 both vertical maps are degreewise weak equivalences, and if $P=A$ degreewise weak equivalences on $s d_{r}(-)^{C_{r}}$.

As to the lower horizontal map, the trace defines a special homotopy inverse to $i$ as follows. We may write out $V\left(\left(T_{n} A\right)_{V},\left(T_{n} P\right)_{V}\right)(x)\left(\right.$ resp. $\left.V\left(\vee_{n} A, \vee_{n} P\right)(x)\right)$ as

$$
\begin{aligned}
& \underset{\substack{(r, s, C) \in \in\left\{n \times n \times \mathcal{Q}^{n}\right\}_{p} \\
r_{i} \geq s_{i-1}}}{\vee} P^{x_{0}}\left(p r_{r_{0}} C_{0}, p r_{s_{p}} C_{p}\right) \wedge \cdots \wedge A^{x_{p}}\left(p r_{r_{p}} C_{p}, p r_{s_{p-1}} C_{p-1}\right) \\
& \left(\text { resp. } \underset{(r, C) \in\left\{n \times \mathscr{C}^{n}\right\}_{p}}{\bigvee} P^{x_{0}}\left(p r_{r_{0}} C_{0}, p r_{r_{0}} C_{p}\right) \wedge \cdots \wedge A^{x_{p}}\left(p r_{r_{p}} C_{p}, p r_{r_{p}} C_{p-1}\right)\right)
\end{aligned}
$$

Note that the condition $r_{i}=s_{i}$ for all $i \in \boldsymbol{Z} /(p+1) \boldsymbol{Z}$ of the trace definition in the previous section becomes $r_{i}=s_{j}$ for all $i, j$ when we include the additional requirement that $r_{i} \geq s_{i-1}$ for all $i$. Thus we define $\operatorname{Tr}: \operatorname{THH}\left(\left(T_{n} A\right)_{V},\left(T_{n} P\right)_{V}\right) \rightarrow T H H\left(\vee_{n} A, \vee_{n} P\right)$ by setting $T r_{x}$ to be the map sending $y$ in the $(r, s, C)$ coordinate to be

$$
\operatorname{Tr}_{x}(y)= \begin{cases}y \text { in the }(r, C) \text { summand } & \text { if all } r_{i} \text { and } s_{j} \text { are equal, } \\ * & \text { otherwise } .\end{cases}
$$

Thus we get for $y$ in the $(r, s, C)$ summand and $z$ in the $(r, C)$ summand that

$$
\begin{aligned}
i_{x} \circ \operatorname{Tr}_{x}(y) & = \begin{cases}y \text { in the }(r, s, C) \text { summand } & \text { if all } r_{i} \text { and } s_{j} \text { are equal, } \\
* & \text { otherwise. }\end{cases} \\
\operatorname{Tr}_{x} \circ i_{x}(y) & = \begin{cases}z \text { in the }(r, C) \text { summand } & \text { if all } r_{i} \text { and } r_{j} \text { are equal, } \\
* & \text { otherwise. }\end{cases}
\end{aligned}
$$

If $\alpha=\left(i_{p} \leftarrow i_{0} \leftarrow i_{1} \leftarrow \cdots \leftarrow i_{p-1} \leftarrow i_{p}\right) \in N_{p}^{c y}(\mathscr{I})$ let

$$
H_{a, x}: V\left(\left(T_{n} A\right)_{V},\left(T_{n} P\right)_{V}\right)(x) \rightarrow V\left(\left(T_{n} A\right)_{V},\left(T_{n} P\right)_{V}\right)(x)
$$

be defined by sending $y$ in the $(r, s, C$ ) summand to

$$
H_{\alpha, x}(y)= \begin{cases}y \text { in the }(r, s, C) \text { summand } & \text { if } i_{k}=0 \Rightarrow r_{k}=s_{k} \\ * & \text { otherwise }\end{cases}
$$

Likewise we define

$$
G_{\alpha, x}: V\left(\left(T_{n} A\right)_{V},\left(T_{n} P\right)_{V}\right)(x) \rightarrow V\left(\left(T_{n} A\right)_{V},\left(T_{n} P\right)_{V}\right)(x)
$$

by sending $z$ in the $(r, C)$ summand to

$$
G_{\alpha, x}(z)= \begin{cases}y_{r} \text { in the }(r, C) \text { summand } & \text { if } i_{k}=0 \Rightarrow r_{k}=r_{k+1}, \\ * & \text { otherwise }\end{cases}
$$

This defines maps $H: V \times N^{\text {cy }}(\mathscr{I}) \rightarrow V$ and $G: W \times N^{c y}(\mathscr{I}) \rightarrow W$. When restricting to $\Delta(1) \subseteq N^{\text {cy }}(\mathscr{I})$ we get that $H$ is a prehomotopy from $i d_{\vee}$ to $i \circ T r$ and $G$ a prehomotopy from $i d_{W}$ to $T r \circ i$. When $P=A$ both are special homotopies.

Hence $V \xrightarrow{i} W$ is a prehomotopy equivalence. If $P_{k}=A_{k}, V \xrightarrow{i} W$ is a special homotopy cquivalence. So all maps in the diagram are weak equivalences, and in case $P=A$ we see that we are in the situation of Lemma 1.5.12 and the result follows.
1.6.21 THH of any ring functor can be calculated by means of an FSP. Given an $\Lambda \subset \mathscr{F} \mathscr{C}$, recall the definition in 1.2 .4 of $\operatorname{FSPs}$ [A] and [A]v. In the same manner we may define for any $A$ bimodule $P$ the $[A]$ (resp. $[A]_{V}$ ) bimodule $[P]$ (resp. $[P]_{V}$ ) given by

$$
[P](X)=\prod_{a \in \mathscr{G}} \bigvee_{b \in \mathscr{C}} P^{X}(a, b) \quad\left(\text { resp. }[P] \vee(X)=\bigvee_{(a, b) \in \mathscr{C}^{2}} P^{X}(a, b)\right)
$$

with the obvious actions. Note that the inclusions $[A]_{\vee} \subseteq[A]$ and $[P]_{\vee} \subseteq[P]$ are not in general stable equivalences. This however is no serious problem, as we have seen in 1.6.10 that $T H H$ may be calculated as the limit of the $T H H$ s obtained by restricting to finite subcategories. Here Blakers-Massey guarantees us stable equivalence, so from now on we may assume that our category has only a finite set of objects.

Now, if $\phi: \mathscr{C} \rightarrow *$ is the functor to the trivial category, we have maps of ring functors on $\mathscr{C} A \rightarrow \phi^{*}[A]_{\vee}$ and $A \rightarrow \phi^{*}[A]$ given by the inclusion $A^{X}(a, b) \subseteq[A]_{\vee}^{X}$ and composition with $[A]_{\vee} \rightarrow[A]$. The map $A \rightarrow \phi^{*}[A]$ is unital if $A$ is. The same maps on bimodules gives us maps


For each $\boldsymbol{x} \in I^{p+1}$, let $j_{x}: V(A, P)(x) \rightarrow V\left([A]_{V},[P]_{V}\right)(x)$ be the map corresponding to $j$. We define a splitting in analogy with the trace (each matrix has only one entry, so this makes sense) $\operatorname{Tr}: \operatorname{THH}\left([A]_{\vee},[P]_{\vee}\right) \rightarrow \operatorname{THH}(A, P)$ given by $\operatorname{Tr}_{x}: V\left([A]_{V}\right.$, $\left.[P]_{V}\right)(x) \rightarrow V(A, P)(x)$ where

$$
p_{x}(y)= \begin{cases}y & \text { if } y \in \operatorname{im} j_{x} \\ * & \text { otherwise }\end{cases}
$$

This is a well defined presimplicial map! (precyclic if $P=A$ ), and to see that it is perhaps best to write out $V\left([A]_{\vee},[P]_{V}\right)(x)$ as follows, again using the natural distributivity between smash and wedge:

$$
V\left([A]_{V},[P]_{V}\right)(x) \cong \bigvee_{(a, b) \in\left\{\mathcal{Q}^{2}\right\}_{p}} P^{x_{0}}\left(a_{0}, b_{p}\right) \wedge \bigwedge_{1 \leq i \leq p} A^{x_{i}}\left(a_{i}, b_{i-1}\right) .
$$

Then $j_{x}$ sends the $c=\left(c_{0}, \ldots, c_{p}\right)$ summand onto the $(c, c)$ summand, and $T r$ sends the ( $a, b$ ) summand onto the basepoint if $a \neq b$ and onto the ( $a_{0}, \ldots, a_{p}$ ) summand if $a=b$. Now, $\operatorname{Tr} \circ j=i d$, and we define a prehomotopy from $j \circ \operatorname{Tr}$ to the identify to get
1.6.22. Lemma. Let $A$ be a ring functor on $\mathscr{C}$ with unit, and let $P$ be a unital $A$ bimodule. Then the inclusion

$$
i: T H H(A, P) \subseteq T H H([A],[P])
$$

is $a$ weak homotopy equivalence, and if $P=A$ a $C$-equivalence.

Proof. Let $X=V=T H H(A, P), Y=T H H([A],[P])$ and $W=T H H\left([A]_{\vee},[P]_{\vee}\right)$, and consider the diagram


We want to show that all maps are weak equivalences, and that if $P=A$ we are in the situation of Lemma 1.5.12.

The right vertical map is induced by a stable equivalence, and so by Lemma 1.6.10 the requirements are satisfied.

Regarding the lower horizontal map, we know that it is split by $\operatorname{Tr}, \operatorname{Tr} \circ j=i d_{V}$, and we define a prehomotopy (special if $P=A$ )

$$
H: T H H\left([A]_{\mathrm{V}},[P]_{\mathrm{V}}\right) \times N^{\mathrm{cy}}(\mathscr{I}) \rightarrow T H H\left([A]_{\mathrm{v}},[P]_{\mathrm{V}}\right)
$$

from $j^{\circ} \operatorname{Tr}$ to the identity as follows. If $\alpha=\left(i_{p} \leftarrow i_{0} \leftarrow i_{1} \leftarrow \cdots \leftarrow i_{p-1} \leftarrow i_{p}\right) \in N_{p}^{\text {cy }}(\mathscr{F})$
 $y \in P^{x_{0}}\left(a_{0}, b_{p}\right) \wedge A^{x_{1}}\left(a_{1}, b_{0}\right) \wedge \cdots \wedge A^{x_{p}}\left(a_{p}, b_{p-1}\right)$ to

$$
H_{\alpha, x}(y)= \begin{cases}y & \text { if for } 0 \leq k \leq p, i_{k}=0 \Rightarrow a_{k}=b_{k} \\ * & \text { otherwise }\end{cases}
$$

Applying $\Omega^{{ }^{x} x}$ and the homotopy colimit this defines the prehomotopy as $p$ varies by restricting to $\Delta(1) \subseteq N^{\text {cy }}(\mathscr{I})$. We note that we get a special homotopy if $P=A$.

## 2. THH of exact categories and algebraic K-theory

### 2.0. Introduction

In this chapter we specialize to the case of the linear ring functor given by $X \mapsto \mathbb{C}(-,-) \otimes \tilde{Z}[X]$ where $\mathbb{C}$ is an additive category. In this case it is possible to say much more about the topological Hochschild homology. In particular the homotopy type may be recognized as the homology of the category itself. Most results will be derived from the mixing of $T H H$ and Waldhausen's $S$ construction, and we prove for split exact categories $\mathbb{C}$, that the inclusion $|T H H(\mathbb{C})| \rightarrow \Omega|T H H(S \mathbb{C})|$ is a $C_{r}$ equivariant homotopy equivalence for every $r$. This means that one could choose to study $\Omega T H H(S-)$ instead for split exact categories, and it may be argued that this perhaps is the right theory even in the more general cases.

The main interest in this theory is as a target for a map from K-theory. For any exact category $\mathbb{C}$ we have an inclusion of simplicial objects

$$
o b S \mathbb{C} \rightarrow T H H_{0}(S \mathbb{C}) \rightarrow T H H(S \mathbb{C}) .
$$

The latter is the inclusion by degeneracies, and the former is given by sending $c \in o b S C$ to $i d_{c} \in S \mathbb{C}(c, c) \subseteq T H H_{0}(S \mathbb{C})$. The obviously maps into the fixed points of both the cyclic actions and the Frobenius maps, and so defines a map

$$
|o b S \mathbb{C}| \rightarrow T C(S \mathbb{C})
$$

analogous to a delooping of the cyclotomic trace [3]. As K-theory depends on the choice of split exact sequences, it is not unreasonable to allow for this in the target. Thus the choice of the model incorporating the $S$ construction should not be considered undesirable. In fact, with this definition we immediately get a spectrum level map from K-theory by applying the $S$ construction repeatedly. Anyway, it is clear that this definition is closer to K-theory at the same time as it has attractive properties inherited from K-theory we do not find in general in the simpler definition. For the special case of the K-theory of a ring we shall see that the above map agrees with earlier definitions. More generally, if $\mathbb{C}$ is any exact category, these maps agree with the Dennis trace when composing with the maps into Hochschild homology.

In the relative situation of a nilpotent extension of a ring, the second author has shown that the map to topological cyclic homology is an equivalence after completion at a prime, thus extending the rational computations of Goodwillie.

As before, we will in this chapter write out the statements in terms of THH only, and leave it to the reader to supply the accompanying statements for $T C$ when necessary.

Roughly, Section 2.1 treats the part of the theory compatible with the cyclic action, whereas the results in Section 2.2 have no cyclic analogues (and thus no counterpart for $T C$ ). The results still have interest in relation to the map from K-theory. Corollary 2.2.4 says that the last map in the definition of the map from K-theory, namely the map by degeneracies suitably stabilized is a homotopy equivalence. More precisely the lower right horizontal map in

is a homotopy equivalence. Thus from the non-cyclic point of view we may choose $\lim _{k \rightarrow \infty} \Omega^{k} T H H_{0}\left(S^{(k)}(\mathbb{C})\right.$ to be our model for topological Hochschild homology. From the point of algebraic K-theory this is a sufficiently simple model to be comparable with K-theory. In fact it was this model which was used to show that stable K-theory is equal to topological Hochschild homology for simplicial rings in [7]. The proof offered here is a simplification of the original argument [7, Theorem 2.6] extended to the present generality. Furthermore, comparison with K-theory makes it possible to translate theorems on K-theory which are sufficiently nice on some endomorphism categories (see 2.3.1). As an example we prove a resolution theorem.
2.0.1. K-theory of exact categories. A category with cofibrations [17] is a pointed category $\mathscr{C}$ together with a cubcategory $\boldsymbol{c o}^{\mathscr{C}}$ satisfying the following axioms:
(1) co $\mathscr{C}$ contains all isomorphism in $\mathscr{C}$.
(2) $c o \mathscr{C}$ contains all maps from the base point.
(3) If $a \rightarrow b \in c \mathscr{C}$ and $a \rightarrow c \in \mathscr{C}$, then the pushout $c \amalg_{a} b$ exists and $c \rightarrow c \amalg_{a} b$ is in co 8.
The morphisms in $c o \mathscr{C}$ will be called cofibrations and typically be represented by a feathered arrow $\mapsto$. A pointed functor between categories with cofibrations is called exact if it preserves cofibrations and the pushout diagrams of axiom (3).

In particular an exact category is a category with cofibration by choosing a zero object and letting the cofibrations to be the admissible monomorphisms.

If $\mathscr{C}$ is a category with cofibrations, a subcategory w $\mathscr{C}$ is a category of weak equivalences in $\mathscr{C}$ if $w \mathscr{C}$ contains all isomorphisms, and if for every commutative diagram

where the horizontal maps to the left are cofibrations and where all vertical arrows are in $w \mathscr{C}$, the induced map $b \coprod_{a} c \rightarrow b^{\prime} \coprod_{a^{\prime}} c^{\prime}$ is also in $w \mathscr{C}$. In this case we call $\mathscr{C}$ a category with weak equivalences and cofibrations. The morphisms in $w \mathscr{C}$ are naturally called weak equivalences.

If $\mathscr{C}$ is a category with cofibrations we have many different choices of weak equivalences. The minimal choice is just the isomorphisms, and the maximal is to allow all morphisms as weak equivalences.

Given a category with cofibrations $\mathscr{C}$, we define a simplicial category with cofibrations $S \mathscr{C}$ as follows. Let [ $n$ ] now denote the ordered set $(0<1<\cdots<n)$ considered as a category, and let $\operatorname{Ar}[n]$ be the corresponding arrow category. We let $S_{n} \mathscr{C}$ be the category of functors $A: A r[n] \rightarrow \mathscr{C}$ having the property that for every $j A(j=j)=0$ and such that if $i \leq j \leq k$ then $A(i \leq j) \rightarrow A(i \leq k)$ is a cofibration and

is a pushout. Thus to give an object in $S_{n} \mathscr{C}$ is the same as to give a sequence of cofibrations

$$
c_{1} \mapsto c_{2} \mapsto \cdots \mapsto c_{n}
$$

together with a choice for each $i \leq j$ of "quotients" $c_{j} / c_{i}$ representing $0 \bigsqcup_{c_{i}} c_{j}$ (for $i=j$ there is no choice but 0 ). The cofibrations in $S_{n} \mathscr{C}$ are natural transformations with
values in $c o \mathscr{C}$, and if $\mathscr{C}$ has a category of weak equivalences $w \mathscr{C}$ so has $S_{n} \mathscr{C}$ by choosing $w S_{\boldsymbol{n}} \mathscr{C}$ to be the natural transformations with values in $w \mathscr{C}$.
The K-theory of a category with cofibrations and weak equivalences $\mathscr{C}$ is then defined to be $\Omega|N w S \mathscr{C}|$ where $N$ is the nerve applied degreewise on the simplicial category $\left\{p \mapsto w S_{p} \mathscr{C}\right\}$. In particular, if $\mathscr{C}$ is an exact category we choose the admissible monomorphisms to be cofibrations and the isomorphisms as weak equivalences. This agrees with the definition of Quillen, and we note that $S \mathscr{C}$ becomes a simplicial exact category with the expected structure. As a last point it should be noted that if $i \mathscr{C} \subseteq \mathscr{C}$ is the isomorphisms, then $o b S \mathscr{C}=N_{0} i S \mathscr{C} \subset N i S \mathscr{C}$ is a homotopy equivalence. When there is no danger of confusion we will simply write $S \mathscr{C}$ for the simplicial set $o b S \mathscr{C}$.
2.0.2. Constructions on linear categories and relations with ring functors. Let $\mathbb{C}$ be a linear category. Then we defined the linear ring functor on $\mathbb{C}$ to be given by $X \mapsto \mathbb{C}(-,-) \otimes \tilde{\mathbf{Z}}[X]$ and we denoted this ring functor again $\mathbb{C}$. For any ring functor we defined the matrix, product, upper triangular matrix of the ring functor. We may also define similar constructions on the category itself, and we will show that as far as topological Hochschild homology concerns, the two approaches are cquivalent. Firstly we see that the linear ring functor on a product of linear categories is isomorphic to the product of the respective linear ring functors. We will consequently make no notational distinctions between these ring functors. As to the matrices, let $m_{n} \mathbb{C}$ be the matrix category of $\mathbb{C}$, i.e. its objects are $n$ tuples of objects in $\mathbb{C}$ and morphisms are $n \times n$ matrices under matrix multiplication. Then

$$
\begin{aligned}
\left(m_{n} \mathbb{C}\right)^{X}(A, B) & =\left(\underset{1 \leq r, s \leq n}{\oplus} \mathbb{C}\left(p r_{r} A, p r_{s} B\right)\right) \otimes \tilde{Z}[X] \\
& \cong \underset{1 \leq r, s \leq n}{\bigoplus}\left(\mathbb{C}\left(p r_{r} A, p r_{s} B\right) \otimes \tilde{Z}[X]\right)=\left(M_{n} \mathbb{C}\right)_{\oplus}^{X}(A, B)
\end{aligned}
$$

and this isomorphism is compatible with composition. Similarly we define the upper triangular matrix category on $\mathbb{C}$, here denoted $t_{\boldsymbol{n}} \mathbb{C}$, and get that

$$
\left(t_{n} \mathbb{C}\right)^{X}(A, B) \cong\left(T_{n} \mathbb{C}\right)_{\oplus}^{X}(A, B)
$$

As $\mathfrak{C}^{n}, m_{n} \mathbb{C}$ and $t_{n} \mathbb{C}$ have the same objects, this means that $\operatorname{THH}\left(m_{n} \mathbb{C}\right)=$ $\left.T H H\left(M_{n} \mathbb{C}\right)_{\oplus}\right)$ and $T H H\left(t_{n} \mathbb{C}\right)=T H H\left(\left(T_{n} \mathbb{C}\right)_{\oplus}\right)$, and as the inclusions $M_{n} \mathbb{C} \rightarrow\left(M_{n} \mathbb{C}\right)_{\oplus}$ and $T_{n} \mathbb{C} \rightarrow\left(T_{n} \mathbb{C}\right)_{\oplus}$ are stable equivalences we have that the induced maps $T H H\left(M_{n} \mathbb{C}\right) \rightarrow T H H\left(m_{n} \mathbb{C}\right)$ and $T H H\left(T_{n} \mathbb{C}\right) \rightarrow T H H\left(t_{n}(\mathbb{C})\right.$ are $C$-equivalences, and hence $C_{r}$ homotopy equivalences for all $r$ on the realizations.

Given any small exact category $\mathbb{C}$, Waldhausen's $S$-construction yields a simplicial additive category $S \mathbb{C}$, and we will study $T H H$ of the ring functors on $S_{p} \mathbb{C}$ given by $X \mapsto S_{p} \mathbb{C}(-,-) \otimes \tilde{Z}[X]$. Call this ring functor for simplicity just $S_{p} \mathbb{C}$. This forms a simplicial object $T H H\left(S_{p} \mathbb{C}\right)$ which we will simply call $T H H(S \mathbb{C})$.
2.0.3. Immediate properties of $T H H(S-)$. We will now specialize, and think of $T H I I$ as a functor from some category of exact categories to pointed cyclic sets. The
methods in [10, Sections 3.4-3.6] apply to any such functor provided it sends the trivial category to a point and respects finite products plus some $\pi_{*}-K$ an condition. The latter point is no problem here as we know by 1.4.7 that THH is degreewise equivalent to a simplicial abelian group. Thus we have for free:

### 2.0.4. Proposition (Additivity). Let © be an exact category. Then

$$
T H H\left(\mathrm{SS}_{2} \mathbb{C}\right) \xrightarrow{d_{0} \times d_{2}} T H H(S \mathbb{C}) \times T H H(S \mathbb{C})
$$

is a special homotopy equivalence.
2.0.5. Proposition (Long exact sequences). Let $F: \mathbb{C} \rightarrow \mathcal{D}$ be an exact functor. Then

$$
\operatorname{THH}(\mathrm{SD}) \rightarrow \operatorname{THH}(\operatorname{SS}(F: \mathbb{C} \rightarrow \mathfrak{D})) \rightarrow T H H(S S C)
$$

is a special fiber sequence (meaning a fibre sequence of cyclic sets such that the leftmost term is specially homotopy equivalent to the actual homotopy fibre).
2.0.6. Proposition (Delooping theorem). Let $\mathbb{C}$ be an exact category. Then

$$
T H H(S \mathbb{C}) \rightarrow \Omega T H H(S S \mathbb{C})
$$

induces an $C_{r}$ equivariant homotopy equivalence on the realization for all $r$.
Thus we have a new spectrum $\left\{k \mapsto T H H\left(S^{k} \mathbb{C}\right)\right\}$ which is an $\Omega$ spectrum after the first term. Just as in the linear case, we shall see that the delooping theorem holds with $\mathbb{C}$ in place of $S \mathbb{C}$ in the case where $\mathbb{C}$ is split exact.
2.0.7. The simplicial theory in an additive category. The following is a collection of special properties of the morphism functor on the additive category and its behaviour on simplicial objects. All results are formal and the reader may safely skip this subsection if he feels familiar with the subject. Let $\mathbb{C}$ be an additive category. Then the simplicial objects in $\mathbb{C}, s \mathbb{C}$, form a closed simplicial model category over the pointed simplicial sets. Most importantly it has products with finite pointed simplicial sets: given $X \in f s_{*} \mathscr{E} n s$ and $c \in s \mathbb{C}$ then their product is $c \otimes \tilde{Z}[X]=\left\{p \mapsto \oplus_{\sigma \in X_{p}-*} c_{p}\right\}$. The simplicial homomorphism group, denoted $s \mathbb{C}(a, b)$, is the simplicial group which in degree $p$ consists of the simplicial $\mathfrak{C}$ maps $a \otimes \tilde{Z}\left[\Delta[p]_{+}\right] \rightarrow b$. The canonical isomorphism

$$
\phi: s \mathbb{C}(a \otimes \tilde{Z}[X], b) \cong s_{*} \mathscr{E} n s(X, s \mathbb{C}(a, b))
$$

is given by $\phi(f)(x \wedge y)(\alpha \otimes z)=f\left(\alpha \otimes z^{*}(x \wedge y)\right)$ where $f \in s \mathbb{C}(a \otimes \tilde{Z}[X], b)_{p}, x \in X_{q}$, $y \in\left(\Delta[p]_{+}\right)_{q}, \quad \alpha \in a_{r}$ and $z \in\left(\Delta[q]_{+}\right)_{r}$ with inverse given by $\phi^{-1}(g)(\beta \otimes$ $(x \wedge y)=g(x \wedge y)(\beta \otimes i d)$ where $g \in s_{*} \mathscr{E} n s(X, s \mathbb{C}(a, b))_{p}, \beta \in a_{q}$ and $i d$ is the identity $[q]=[q] \in \Delta[q]$.

The inclusion

$$
s \mathbb{C}(a, b) \rightarrow s \mathbb{C}(a \otimes \tilde{Z}[X], b \otimes \tilde{Z}[X])
$$

given by $f \mapsto f \otimes$ id is compatible with composition, and in the special case $X=S^{n}$ it is a homotopy equivalence as seen by induction as follows. It is enough to consider the case $X=S^{1}$. The statement then is equivalent to showing that $s_{*} \mathscr{E} n s\left(S^{1},-\right)$ and $-\otimes \tilde{Z}\left[S^{1}\right]$ are loop and suspension (bar construction) on simplicial abelian groups, and that the inclusion into the loops of the suspension is a homotopy equivalence, which is shown in [13, II, 6.4]. That the map is as given follows from the diagram with short exact rows

where $G$ is any simplicial abelian group. Here $i_{1}$ is the obvious inclusion, $p$ the projection $\Delta[1] \rightarrow S^{1}$ and $e v_{1}$ is the evaluation at $1 \in \Delta[1]$. The maps $u$ and $v$ are defined as follows. Let $h: \Delta[1] \times \Delta[1] \rightarrow \Delta[1]$ be the construction sending a pair onto the one with the most zeros. Then if $g \in G_{p}$ and $\sigma \in \Delta[1]_{p}$ we define $v(g, \sigma)$ : $\Delta[1] \wedge \Delta[p]_{+} \rightarrow G \otimes \tilde{Z}\left[S^{1}\right]$ to be the map sending $\tau \wedge \omega$ to $\omega^{*} g \otimes p \circ h(\sigma, \tau)$. We see that this makes the right square commute, and induces $u(g)(\tau \wedge \omega)=\omega^{*} g \otimes \tau$. Setting $G=s \mathbb{C}(a, b)$ and composing with the canonical isomorphism, we see that the map coincides with the inclusion above.

An important special feature is that

$$
s \mathbb{C}(a, b) \otimes \tilde{Z}[X] \cong s \mathbb{C}(a, b \otimes \tilde{Z}[X])
$$

as a sum (=product!) of morphisms is the same as a morphism to the sum.
If $a \in \mathbb{C}$ considered as a constant simplicial object, and $b \in s \mathbb{C}$ then $\mathfrak{C}(a, b)=\left\{p \mapsto \mathbb{C}\left(a, b_{p}\right)\right\}$ may naturally be identified with $s \mathbb{C}(a, b)$, so in particular if $a, b \in \mathbb{C}$ and $X \in f s_{*} \tilde{E} n s$ then $\mathbb{C}(a, b) \otimes \tilde{\boldsymbol{Z}}[X] \cong s \mathcal{C}(a, b \otimes \tilde{Z}[X])$. All in all we get that the inclusion

$$
\mathfrak{C}(a, b) \otimes \tilde{Z}[X] \rightarrow s \mathbb{C}\left(a \otimes \tilde{Z}\left[S^{n}\right], b \otimes \tilde{Z}\left[X \wedge S^{n}\right]\right)
$$

is a homotopy equivalence compatible with the compositions.
2.0.8. Ring functors on non-linear categories. We end this section by taking a quick and incomplete look at how to extend the idea of the linear ring functor to other categories with weak equivalences and cofibrations. Note that a category with cofibrations $\mathscr{C}$ has finite sums. For a finite set $X$ and an object $c \in \mathscr{C}$ let $V_{X} c$ denote the sum (over the basepoint 0 ) with one copy of $c$ for each element in $X$. That means that we can define a product with finite pointed simplicial sets by setting for $c \in s \mathscr{C}$
and $X \in s_{*} \mathscr{E} n s$

$$
(c \otimes X)=\left\{p \mapsto \bigvee_{x_{p}-*} c_{p}\right\}
$$

(the sum of one copy of $c_{p}$ for each non-basepoint of $X_{p}$ plus a copy of the base-point). We then define the functor

$$
F: f s_{*} \mathscr{E} n s \rightarrow s_{*} \mathscr{E} n s^{\mathscr{C}^{\circ} \times \mathscr{C}}
$$

by

$$
F^{X}(a, b)=s \mathscr{C}(a, b \otimes X)
$$

or some fibrant rectification of this in case we have trouble with homotopy liftings. This functor satisfies all the required commutative diagrams for being a ring functor on $\mathscr{C}$. The only problem is the connectivity requirements. So we have to require that for $n$-connected $X$ and any $a, b \in \mathscr{C}$ :
(1) $s \mathscr{C}(a, b \otimes X)$ is $n$ connected (or more generally $n-d(a)+c(b)$ connected),
(2) the map $S^{1} \wedge s^{1} \mathscr{C}(a, b \otimes X) \rightarrow s_{\mathscr{C}}\left(a, b \otimes\left(S^{1} \wedge X\right)\right) \rightarrow s_{\mathscr{C}}\left(a, b \otimes S^{1} \wedge X\right)$ is $2 n-c$ connected for some $c$ not depending on $X$.
In the exact case, tensor product moves outside the morphisms, and so the connectivity requirements are trivially met. However, in general there seems to be trouble. An important case where we still are able to meet the connectivity requirements is the following:
2.0.9. The space case. Let $\mathscr{C}=R_{f}(X)$; the category of spaces with a given space $X$ as a retract, and only finitely many cells outside $X$. For any $Y \in \mathscr{C}$ and finite pointed simplicial set $V$ we define $Y \otimes V$ to be the cofibre of $Y \times|*| \cup X \times|V| \rightarrow Y \times|V|$ (which is the realization of the construction above). Let $s \mathscr{C}(-,-)$ denote the simplicial mapping space of spaces over $X$, and define our ring functor on $\mathscr{C}$ to be

$$
F^{V}(Y, Z)=s^{\mathscr{B}}(Y, Z \otimes V)
$$

This meets the relaxed connectedness conditions if we set $d^{F}(Y)$ to be the dimension of $Y$ over $X$ and $c^{F}(Z)$ to be the connectivity of $X \rightarrow Z$.

### 2.1. Cofinality, split exact categories and Morita equivalence for rings

In this section we prove two theorems with a distinctive K-theoretic flavor. The first shows that the cofinality result of K-theory is true in topological Hochschild homology. The proof consists of simply displaying a special homotopy. The second shows that if $\mathbb{C}$ is a split exact category, then the map $\mathbb{C} \rightarrow \Omega S \mathbb{C}$ induces $C_{r}$, equivariant homotopy equivalences on $T H H$. In general, by the delooping theorem we have a spectrum $n \mapsto \operatorname{THH}\left(S^{(n)} \mathbb{C}\right)$ (where $S^{(n)}(\mathbb{C})$ is the $S$ construction applied $n$ times to $\mathbb{C}$ ) which is an $\Omega$-spectrum after the first term. The proposition then tells us that for split
exact category this spectrum is an $\Omega$-spectrum at all places. This is in analogy with the matrix spectrum which was used to define the $\Gamma$ space structure in [2]. We end the section with a discussion of the map from K-theory and show that it agrees with the one defined in [2].

Let $\mathbb{C} \subseteq \mathfrak{D}$ be a full inclusion of additive categories. We say that $\mathbb{C}$ is cofinal in $\mathfrak{D}$ if for any $d \in \mathfrak{D}$ there exists some $d^{\prime} \in \mathfrak{D}$ such that $d \oplus d^{\prime} \in \mathbb{C}$.
2.1.1. Lemma (Cofinality). Let $j: \mathbb{C} \subset \mathfrak{D}$ be an inclusion of a cofinal subcategory into an additive category. Then

$$
T H H(\mathbb{C}) \rightarrow T H H(D)
$$

is a special homotopy equivalence.

Proof. For each $d \in \mathfrak{D}$ choose a $c(d) \in \mathbb{C}$ such that $d$ is a summand in $c(d)$, and if $d$ actually is in $\mathbb{C}$, choose $c(d)=d$. Let $d \xrightarrow{i(d)} c(d) \xrightarrow{p(d)} d$ be the chosen inclusion and projection into and from the sum. Then

$$
\bigvee_{\left(d_{0} \ldots, d_{p}\right) \in \mathbb{D}^{p+1}} \mathcal{D}\left(p\left(d_{0}\right), i\left(d_{p}\right)\right) \otimes \tilde{Z}\left[S^{x_{0}}\right] \wedge \cdots \wedge \mathfrak{D}\left(p\left(d_{p}\right), i\left(d_{p-1}\right)\right) \otimes \tilde{Z}\left[S^{x_{p}}\right]
$$

is a map $V(\mathcal{D})(\boldsymbol{x}) \rightarrow V(\mathbb{C})(\boldsymbol{x})$. This map is compatible with the cyclic operations and hence defines a map $D(p, i): T H H(\mathbb{D}) \rightarrow T H H(\mathbb{C})$. Obviously $D(p, i) \circ T H H(j)$ is the identity on $T H H(\mathbb{C})$ and we will show that the other composite is specially homotopic to the identity. The desired special homotopy can be expressed as follows. Let $\alpha=\left(i_{p} \leftarrow i_{0} \leftarrow \cdots \leftarrow i_{p-1} \leftarrow i_{p}\right) \in N^{c y}(\mathscr{F})$ and let $\left.p(d)^{i_{t}}\right)\left(\right.$ resp. $\left.i(d)^{i_{k}}\right)$ be $p(d)$ (resp. $\left.i(d)\right)$ if $i_{k}=1$ and the identity on $d$ otherwise. Then the desired special homotopy $T H H(D) \times$ $N^{c y}(\mathscr{I}) \rightarrow T H H(D)$ is

$$
\begin{aligned}
& \underset{x \in I^{p+1}}{\operatorname{holim}} \Omega^{\sqcup x} \bigvee_{\left(d_{0}, \ldots, d_{p}\right) \in \mathfrak{D}^{p+1}} \mathcal{D}\left(p\left(d_{0}\right)^{i_{0}}, i\left(d_{p}\right)^{i_{0}}\right) \otimes \tilde{Z}\left[S^{x_{0}}\right] \\
& \wedge \cdots \wedge \mathfrak{D}\left(p\left(d_{p}\right)^{i_{p}}, i\left(d_{p-1}\right)^{i_{p-1}}\right) \otimes \tilde{Z}\left[S^{x_{p}}\right] .
\end{aligned}
$$

2.1.2. A delooping in the split exact case. An exact category $\mathfrak{C}$ is called split if all exact sequences split. The category of finitely generated projective modules is an example of a split category, and generally, any additive category may be regarded as an exact category by considering only the split exact sequences.
2.1.3. Proposition. Let © be a split exact category. Then there is a $C_{r}$ equivariant homotopy equivalence

$$
|T H H(\mathbb{C})| \xrightarrow{\cong} \Omega|T H H(S \mathbb{C})|
$$

Proof. As $\mathbb{C}$ is split we have by [10] that for each $n$ the functor $t_{n} \mathbb{C} \rightarrow S_{n} \mathbb{C}$ given by sending $\left(c_{1}, \ldots, c_{n}\right)$ to $c_{1} \mapsto c_{1} \oplus c_{2} \mapsto \cdots \rightarrow c_{1} \oplus \cdots \oplus c_{n}$ is an equivalence. Let $S_{n} \mathbb{C} \rightarrow \mathbb{C}^{n}$ be the functor sending $c_{1} \mapsto c_{2} \mapsto \cdots \mapsto c_{n}$ (with choices of quotients) to
$\left(c_{1}, c_{2} / c_{1}, \ldots, c_{n} / c_{n-1}\right)$. The linear ring functor associated to the $\mathbb{C}^{n}$ and the $n$ fold product of the linear ring functor associated to $\mathbb{C}$ are isomorphic, and both interpretations will again be denoted $\mathbb{C}^{n}$.

Consider the commutative diagram


The labelled arrows are $C$-equivalences for the given reasons (and hence induce $C_{r}$ equivariant homotopy equivalences for all $r$ on the realizations). Given a simplicial object $X$, let $P X$ be the path space with $P X_{n}=X_{n+1}$ and face and degeneracy maps shifted up by one. The sequence $\mathbb{C} \rightarrow P S \mathbb{C} \rightarrow S \mathbb{C}$ yields a diagram for each $n:$


The vertical maps are special homotopy equivalences and the lower sequence is the trivial fibration. The bisimplicial sets involved satisfy the $\pi_{*}-$ Kan condition (see [4]) as they are related by termwise (ordinary) equivalences to bisimplicial abelian groups by the results in 1.4.7. Thus we know that the total of the upper row is a cyclic fibre sequence. Furthermore $P S C$ is contractible by exact functors, so $T H H(P S \mathbb{C})$ is contractible as a cyclic space and the result follows.

The above proof breaks down in the cases where exact sequences which do not split are allowed. It may still be argued that $\Omega|\operatorname{THH}(S \mathbb{C})|$ is a better behaved theory than $T H H(\mathbb{C})$ in view of the properties listed in Section 2.0. It would be interesting to do calculations on concrete examples to gauge the difference between these theories. In Section 2.3.4 we will give a simple example.

By forgetting structure, any exact category $\mathbb{C}$ is an additive category, and choosing the split monomorphisms to be cofibrations we define the K-theory of any additive category. We write $\bar{S} \mathbb{C}$ for the $S$ construction applied to the underlying additive (split exact) category. Thus any exact category give rise to two spectra, namely $\left\{n \mapsto \operatorname{THH}\left(\bar{S}^{(n)} \mathfrak{C}\right)\right\}$ and $\left\{n \mapsto T H H\left(S^{(n)}(\mathbb{C})\right\}\right.$. The former is always an $\Omega$-spectrum, and is the translation of the matrix spectrum of [2], whereas the other is dependent upon the exact sequences in the category.
2.1.4. The projective modules over a ring. Let $A$ be an associative ring with unit and let $\mathscr{P}_{A}$ be the category of finitely generated projective $A$ modules. $A$ itself may be
considered as a full subcategory whose only object is the rank one module $A$, and each element $a \in A$ identified with the homomorphism given by multiplication by $a$.

The classical Morita equivalence now follows from cofinality. One should note that given two Morita equivalent rings considered as categories with only one object, their homologies do not coincide. In particular, the homology of the ring considered as a category with one object is different from the homology of the category of finitely generated projective modules (which is a good thing as $H_{n}(A, A)=H H_{n}(Z[A], A)$ is not topological Hochschild homology).
2.1.5. Proposition (Morita equivalence for rings). Let $A$ be an associative ring with unit. Then the inclusion $A \subseteq \mathscr{P}_{A}$ given by sending the same object of $A$ to the rank one module induces a special homotopy equivalence

$$
T H H(A) \xrightarrow{\simeq} T H H\left(\mathscr{P}_{A}\right) .
$$

Proof. Let $\mathscr{F}_{A}$ be the category of finitely generated free modules, and let $\mathscr{F}_{A}^{k}$ be the subcategory of free modules of rank less than or equal to $k$. Then the inclusion $m_{k} A \rightarrow \mathscr{F}_{A}^{k}$, given by regarding $m_{k} A$ as the subcategory with only object the rank $k$ module is an equivalence of categories. Consider the diagram where the limit is taken with respect to inclusion by zeros:


The maps are $C$-equivalences for the given reasons and the result follows.
2.1.6. The map from K-theory and agreement with earlier definitions in the ring case. As remarked earlier, the raison d'etre for topological Hochschild homology is that it is the target for a map from algebraic K-theory. In our setting this map is extremely simple, and we show that it agrees with the existing definition in the case of rings. We note that the image of this map consists of fixed points under both the cyclic action and the Frobenius map and so the map factors through topological cyclic homology.

Consider the map $D: o b \mathbb{C} \rightarrow T H H_{0}(\mathbb{C}) \rightarrow T H H(\mathbb{C})$ given by sending an object $c \in o b \mathbb{C}$ to the corresponding identity morphism in

$$
\mathfrak{C}(c, c) \subseteq \underset{x \in I}{\text { holim }} \Omega^{x} \bigvee_{c \in \mathbb{C}} \mathbb{C}(c, c) \otimes \tilde{\boldsymbol{Z}}\left[S^{x}\right]=T H H_{0}(\mathbb{C})
$$

followed by the inclusion by degeneracies. This immediately gives the map from algebraic K-theory by using $S \mathbb{C}$ as our exact category:

$$
\Omega|S D|: \Omega|o b S \mathbb{C}| \rightarrow \Omega|T H H(S \mathbb{C})| .
$$

One should note that the observation that this map induces a map of spectra is trivial, for the commutativity of the diagram

follows from the definition of $D$ applied to the sequences


For any additive category let $i \mathbb{C}$ be the subcategory of isomorphisms, and let Ni denote the nerve of $i \mathbb{C}$. The inclusion $o b \mathbb{C}=N_{0} i \mathbb{C} \rightarrow N i \mathbb{C}$ is a homotopy equivalence. There is a map

$$
N i \mathbb{C} \rightarrow N^{c y}(\mathbb{C}) \rightarrow T H H(\mathbb{C})
$$

given by sending an object

$$
c_{0} \stackrel{\alpha_{1}}{\cong} c_{1} \stackrel{\alpha_{2}}{\cong} \cdots \stackrel{a_{p}}{\cong} c_{p} \in N_{p} i \mathbb{C}
$$

to

$$
c_{p} \stackrel{\left(\prod_{i=1}^{p} a_{i}\right)^{-1}}{\longleftrightarrow} c_{0} \stackrel{\alpha_{1}}{\longleftrightarrow} \cdots \stackrel{\alpha_{p}}{\longleftrightarrow} c_{p} \in N^{\mathrm{cy}}(\mathbb{C})
$$

which is sent to THH as a map from an appropriate smash product of $S^{0}$ with itself. Now, $D$ factors through these maps


The cyclotomic trace from K-theory of a ring to topological Hochschild homology of a ring $A$ is defined in [3] to be the plus construction on the realization of the map

$$
\underset{k}{\lim } N i\left(m_{k} A\right) \rightarrow \underset{k}{\lim } T H H\left(m_{k} A\right)
$$

(or more precisely, in [3] it is the map of $\Gamma$-spaces arising from this map, and sent onto the limit of the fixed points of the finite actions). If we compose with the equivalences $\lim _{k \rightarrow \infty} T H H\left(m_{k} A\right) \xrightarrow{\simeq} T H H\left(\mathscr{P}_{A}\right) \xrightarrow{\simeq} \Omega T H H\left(S \mathscr{P}_{A}\right)$ we see that we have a factorization


By the universality of the plus construction with respect to mappings into spaces with fundamental groups void of perfect subgroups the lower map must realize the map defined by the plus construction. So composing with the equivalence תobSC $\xrightarrow{\simeq} \Omega N i S \mathbb{C}$ we get that

$$
\Omega S D: \Omega S \mathscr{P}_{A} \rightarrow \Omega T H H\left(S \mathscr{P}_{A}\right)
$$

agrees with Bökstedt's trace. Inverting $\Omega T H H\left(S \mathscr{P}_{A}\right) \cong T H H\left(\mathscr{P}_{A}\right) \cong T H H(A)$ we may regard this as a map $\Omega o b S \mathscr{P}_{A} \rightarrow T H H(A)$.

As earlier observed, $S D: o b S C \rightarrow T H H(S C)$ maps into the fixed points of both the cyclic action and the Frobenius maps. This is so because in each degree an element $c \in S \mathbb{C}$ is mapped onto the point represented by an appropriate wedge of $S^{0}$ 's mapping onto $c=c=\cdots=c \in S \mathbb{C}(c, c) \wedge \cdots \wedge S \mathbb{C}(c, c)$. Thus we have a lifting to the topological cyclic homology agreeing with the one defined in [3] after completion at a prime.
2.1.7. Weak equivalences. So far, the only weak equivalences we have considered are the isomorphisms. However, at this level, the introduction of weak equivalences poses no new problems. Let $w \mathbb{C}$ be a subcategory of weak equivalences in an exact category $\mathfrak{C}$ (for instance homotopy equivalence between simplicial modules of some sort). Let $N w S \mathbb{C}$ be the bisimplicial category with objects elements in the ordinary nerve of $w S \mathbb{C}$ and morphisms natural transformations in $S \mathbb{C}$ (note that all morphisms in $S \mathbb{C}$ are allowed, not just the weak equivalences). This becomes a new exact category. Thus, using the linear ring functor construction again we define the topological Hochschild homology of $\mathbb{C}$ with respect to $w \mathbb{C}$ as $T H H(N w S \mathbb{C})$. This is compatible with the earlier definition in the case of isomorphisms as the proof of Waldhausen ([17, 1.4.1], extended to allow for morphisms) that $S \mathbb{C} \simeq N i S \mathbb{C}$ only required a simplicial homotopy of exact categories and hence induces a homotopy equivalence of cyclic objects $T H H(S \mathbb{C}) \simeq T H H(N i S \mathbb{C})$. Thus, if $\mathbb{C}$ is an exact category, one may equally well map K-theory of the isomorphisms to THH via

$$
o b N i S \mathbb{C} \rightarrow T H H_{0}(N i S \mathbb{C}) \subseteq T H H(N i S \mathbb{C})
$$

which is compatible with the homotopy equivalence $S \mathbb{C} \subseteq N i S \mathbb{C}$.

More generally, if $w \mathbb{C}$ is any subcategory of weak equivalences in an exact category we can define the map from the algebraic K-theory $\Omega|o b N w S \mathbb{C}|$ to $\Omega|T H H(N w S \mathbb{C})|$ by the same map as before (inclusion by the identity followed by the degeneracies). In the case where $w \mathbb{C}$ are the isomorphisms, this brings nothing new, but if it is something else $\Omega|T H H(N w S \mathbb{C})|$ is a new theory, potentially reflecting the algebraic K-theory better than just $\operatorname{THH(C)}$. We have not needed the result, and thus not carried through the calculations, but it seems reasonable to believe that for exact categories where the admissible monomorphisms split up to weak equivalence, an analysis of the proof of the equivalence of the $S$ construction and the $\Gamma$ category machine of Segal would prove a delooping theorem for this construction. One outcome would then be that in these cases this definition would agree with earlier ones on simplicial rings and that the trace map could have been constructed directly.

### 2.2. A non-cyclic reduction in the case of the K-theory of an exact category and comparison with the homology of categories

In this section we will give a simpler, but not cyclic, description of the topological Hochschild homology of an exact category. This reduction, together with the treatment of the split exact categories in the previous section, gives that for additive categories THH may be reduced to the ordinary homology in the sense of BauesWirsching. The reduction is a parallel to the reduction in [7], and shows that the non-cyclic models there agree with the present definition. When our category is the category of finitely generated projective modules over a ring the result is not new. Together with Morita equivalence for $T H H$ this gives a direct proof of the main result in [12].

Even though the reduction does not preserve the cyclic structure, it is still of general interest because the map form K-theory factors through the simpler theory. In particular, the result in [7] was obtained using this model, a fact that is important even when one wants to include the cyclic structure into the picture.
2.2.1. The simplicial abelian group $\boldsymbol{R}(\mathbb{C})$. In Section 1.4 .7 we showed that $T H H$ of ring functors with a "linear" bimodule may be rewritten as the bisimplicial abelian group:

$$
\begin{aligned}
& \operatorname{THH}(A, P) \simeq \simeq p \mapsto \underset{x \in I^{p+1}}{\operatorname{holim}} \Omega^{\Delta x} \underset{\left(c_{0}, \ldots, c_{p}\right) \in \mathbb{C}^{p+1}}{ } P^{x_{0}}\left(c_{0}, c_{p}\right) \otimes \tilde{Z}\left[A^{x_{1}}\left(c_{1}, c_{0}\right)\right] \\
&\left.\otimes \cdots \otimes \tilde{Z}\left[A^{x_{p}}\left(c_{p}, c_{p-1}\right)\right]\right\}
\end{aligned}
$$

Let $\mathbb{C}$ bc an additive category. Then in particular it is a linear category, and again, we call the ring functor induced by $X \mapsto(\mathbb{C}(-,-) \otimes \tilde{Z}[X]$ simply $\mathbb{C}$. By the discussion
in 2.0 .7 we have an inclusion which is a homotopy equivalence

$$
\mathfrak{C}(a, b) \otimes \tilde{Z}\left[S^{k}\right] \rightarrow s \mathbb{C}\left(a \otimes \tilde{Z}\left[S^{n}\right], b \otimes \tilde{Z}\left[S^{n+k}\right]\right)
$$

compatible with composition. Hence we may rewrite the above formula in this particular case as

$$
\operatorname{THH}(\mathbb{C}) \simeq R(\mathbb{C})=\left\{p \mapsto \underset{x \in I^{p+1}}{\text { holim }} \Omega_{\left(c_{0}, \ldots, c_{p}\right) \in \mathscr{P}^{p+1}}^{\mathrm{Lx}_{x}} \underset{(\mathbb{C})(x)}{\bigoplus_{\mathrm{C}}}\right\}
$$

where $W(\mathbb{C})(x)$ denotes

$$
\begin{aligned}
s \tilde{C}\left(c_{0} \otimes \tilde{Z}\left[S^{x_{1}+\cdots+x_{p}}\right],\right. & \left.c_{p} \otimes \tilde{Z}\left[S^{\llcorner x}\right]\right) \otimes \tilde{\boldsymbol{Z}}\left[s \mathbb { C } \left(c_{1} \otimes \tilde{Z}\left[S^{x_{2}+\cdots+x_{p}}\right]\right.\right. \\
& \left.\left.c_{0} \otimes \tilde{\boldsymbol{Z}}\left[S^{x_{1}+\cdots+x_{p}}\right]\right)\right] \otimes \cdots \otimes \tilde{Z}\left[s \mathbb{C}\left(c_{p}, c_{p-1} \otimes \tilde{Z}\left[S^{x_{p}}\right]\right)\right] .
\end{aligned}
$$

One nice aspect about this representation is the ability to express elements of $W(\mathbb{C})(x)$ as linear combinations of sequences of simplicial "maps" of the form

$$
a_{-1} \stackrel{\alpha_{0}}{\leftrightarrows} a_{0} \stackrel{\alpha_{1}}{\leftrightarrows} a_{1} \stackrel{\alpha_{2}}{\leftrightarrows} \cdots \stackrel{a_{p-1}}{\stackrel{a}{p-1}} \stackrel{a_{p}}{\leftrightarrows} a_{p}
$$

where $a_{-1}=c_{p} \otimes \tilde{Z}\left[S^{\sqcup x}\right], a_{k}=c_{k} \otimes \tilde{Z}\left[S^{x_{k+1}+\cdots+x_{p}}\right]$ and $\alpha_{i} \in s \mathscr{C}\left(a_{i}, a_{i-1}\right)$. Note however that sums of maps beyond $a_{-1} \leftarrow a_{0}$ are not sums of maps in $s \mathbb{C}$, but in $\tilde{\boldsymbol{Z}} s \mathbb{C}$.

With our new model we get that $\mathbb{C} \mapsto R(\mathbb{C})$ is a functor into simplicial abelian groups satisfying $R(0)=0$ and $R(\mathbb{C} \times \mathcal{D}) \xrightarrow{\simeq} R(\mathbb{C}) \times R(\mathcal{D})$ (see Section 1.6.14), and so by [7, Lemma 2.5] we have that $d_{0}+d_{2} \simeq d_{1}: R\left(S S_{2} \mathbb{C}\right) \rightarrow R(S \mathbb{C})$ where $d_{0}, d_{1}, d_{2}: S_{2} \mathbb{C} \rightarrow S_{1} \mathbb{C} \cong \mathbb{C}$ are the face maps in the $S$ construction, sending a short exact sequence $0 \rightarrow C^{\prime} \rightarrow C \rightarrow C^{\prime \prime} \rightarrow 0$ onto $C^{\prime \prime}, C$ and $C^{\prime}$ respectively. This is not true for each $R_{q}$. The same problem is present in the proof of Theorem 2.6 in [7] and we are grateful to Lannes and Oliver for pointing this out. However, if we stabilize in the $S$ direction we get

$$
d_{0}+d_{2} \simeq d_{1}: \lim _{k \rightarrow \infty} \Omega^{k} R\left(S^{(k)} S_{2} \mathbb{C}\right) \rightarrow \lim _{k \rightarrow \infty} \Omega^{k} R\left(S^{(k)} \mathbb{C}\right)
$$

which clearly suffice for the applications in [7]. This follows from [7, Lemma 2.5] and
2.2.2. Lemma. Let $Y$ be a functor from exact categories to simplicial abelian groups such that $Y(0)=0$. Then

$$
d_{0}+d_{2} \simeq d_{1}: \lim _{k \rightarrow \infty} \Omega^{k} Y\left(S^{(k)} S_{2} \mathfrak{C}\right) \rightarrow \lim _{k \rightarrow \infty} \Omega^{k} Y\left(S^{(k)}(\mathbb{C})\right.
$$

Proof. The only thing we need to show is that $\lim _{k \rightarrow \infty} \Omega^{k} Y S^{(k)}$ respects products, but this is easy: regarded as a map of $2 k$ multisimplicial objects the (split) map

$$
Y\left(S^{(k)}\left(\mathbb{C} \times S^{(k)} \mathfrak{D}\right) \rightarrow Y\left(S^{(k)} \mathfrak{C}\right) \times Y\left(S^{(k)} \mathfrak{D}\right)\right.
$$

is an isomorphism for total degree less than $2 k$, and hence is $2 k$ connected.
2.2.3. Proposition. Let $\mathbb{C}$ be an exact category, and let $s: T H H_{0} \rightarrow T H H_{p}$ be the zeroth degeneracy applied $p$ times. Then

$$
s: \lim _{k \rightarrow \infty} \Omega^{k} T H H_{0}\left(S^{(k)} \mathbb{C}\right) \subseteq \lim _{k \rightarrow \infty} \Omega^{k} T H H_{p}\left(S^{(k)} \mathbb{C}\right)
$$

is a homotopy equivalence.
Proof. We know that $T H H(-) \rightarrow R(-)$ is a degreewise equivalence, and so it is enough to show the proposition on $R$. Let $\Re=\lim _{k \rightarrow \infty} \Omega^{k} R S^{(k)}$. Let $d: \Re_{p} \rightarrow \Re_{0}$ be the zeroth face map applied $p$ times. Then $d \circ s=i d_{\mathfrak{F}_{0}}$. We want to show $s \circ d: \Re_{p}(\mathbb{C}) \rightarrow \mathfrak{R}_{p}(\mathbb{C})$ is homotopic to the identity. Now, if $\boldsymbol{x} \in I^{p+1}$, then

$$
s_{x} \circ d_{x}: W(\mathcal{D})(x) \rightarrow W(\mathcal{D})(x)
$$

sends

$$
a=\left(a_{-1} \stackrel{\alpha_{0}}{\leftarrow} a_{0} \stackrel{\alpha_{1}}{\leftarrow} a_{1} \stackrel{\alpha_{2}}{\leftarrow} \cdots \stackrel{\alpha_{p-1}}{\leftarrow} a_{p-1} \stackrel{\alpha_{p}}{\leftarrow} a_{p}\right)
$$

to $a_{-1} \stackrel{\beta}{\longleftrightarrow} a_{p}=a_{p}=\cdots=a_{p}$ where $\beta=\alpha_{0} \circ \alpha_{1} \circ \cdots \circ \alpha_{p}$.
Proceedings in analogy with [7, Theorem 2.6] we define two natural transformations $T_{i d}, \quad T_{\beta}: \Re_{p} \rightarrow \Re_{p} S_{2}$ induced by transformations $t_{i d}, t_{\beta}: W(\mathcal{D})(x) \rightarrow$ $W\left(S_{2} \mathcal{D}\right)(x)$ where $\mathfrak{D}$ is an exact category and $x \in I^{p+1}$. Let $a \in W(\mathcal{D})(x)$ as above, and let $\beta_{k}=\alpha_{k} \circ \cdots \circ \alpha_{p}$ with $\beta=\beta_{0}$. Then

and

where $i_{j}$ is the $j$ th inclusion, $\pi_{j}$ the $j$ th projection and $\Delta$ the diagonal.

Again we have the relations (with the $d_{i}$ being the face maps $S_{2} \mathfrak{D} \rightarrow \mathcal{D}$ defined above).

$$
d_{0} T_{i d}=i d, \quad d_{2} T_{\beta}=s_{x} \circ d_{x}, \quad d_{2} T_{i d}=d_{0} T_{\beta}=0, \text { and } d_{1} T_{i d}=d_{1} T_{\beta}
$$

and using the formula $d_{0}+d_{2} \simeq d_{1}$ just before Lemma 2.2 .2 we get that

$$
i d=d_{0} T_{i d} \simeq d_{1} T_{i d}=d_{1} T_{\beta} \simeq d_{2} T_{B}=S_{x} \circ d_{x}
$$

By the realization lemma and 2.0 .6 this implies
2.2.4. Corollary. $\lim _{k \rightarrow \infty} \Omega^{k} T H H_{0}\left(S^{(k)} \mathbb{C}\right) \subseteq \lim _{k \rightarrow \infty} \Omega^{k} T H H\left(S^{(k)} \mathbb{C}\right) \approx \Omega T H H(S \mathbb{C})$ are homotopy equivalences.

Note that $\mathfrak{R}_{0}=\lim _{k \rightarrow \infty} \Omega^{k} R_{0} S^{(k)}$ is a particularly simple object, namely

$$
\Re_{0}(\mathbb{C})=\lim _{k \rightarrow \infty} \Omega^{k} \underset{x \in I}{\text { holim }} \Omega^{x} \bigoplus_{c \in S^{(k)} \mathbb{C}} S^{(k)} \mathbb{C}(c, c) \otimes \tilde{Z}\left[S^{x}\right] \simeq \lim _{k \rightarrow \infty} \Omega^{k} \bigoplus_{c \in S^{(k)} \mathbb{C}} S^{(k)} \mathbb{C}(c, c)
$$

which, by a closely analogous proof to the above, was shown in [7] to be equivalent to what there was called the topological Hochschild homology of an exact category.

Perhaps the diagrams in the proof above can be understood a bit better if we forget some structure. For $T_{i d}$ we basically have the information

where we have supressed a top row

$$
0 \leftarrow a_{p}=a_{p}=\cdots=a_{p} \leftarrow 0
$$

If the upper row in the diagram represented a simplicial map (in the THH direction), say $T: T H H(\mathbb{C}) \rightarrow T H H(\mathbb{C})$, this would have represented a simplicial homotopy from $T$ to the identity. Similarly, the second diagram carries the information

where we have supressed the bottom row

$$
0 \leftarrow a_{0} \stackrel{a_{1}}{\leftrightarrows} a_{1} \stackrel{a_{2}}{\leftrightarrows} \cdots \stackrel{\alpha_{p-1}}{\stackrel{ }{2}} a_{p-1} \stackrel{\alpha_{p}}{\leftrightarrows} 0 .
$$

If both the rows $\left(s_{x} \circ d_{x}(a)\right.$ and $T_{x}(a)$ respectively) represented simplicial maps this would have given us a homotopy from $s \circ d$ to $T$, and so composed with the homotopy above given us a homotopy from the identity to $s \circ d$ proving that for any additive category $\mathbb{C}$ topological Hochschild homology was concentrated in degree zero. Of course, this is not the case: none of the proposed maps are simplicial (except the identity), and it is crucial to use the additivity coming from the $S$ constructions so that we can forget the irrelevant rows and get homotopies, not in the THH direction, but in the $S$ direction.

Let $\mathbb{C}$ be a linear category. Then the simplicial abelian group $F . \mathbb{C} \cong C N(\mathbb{Z}, \mathbb{C})$ of [7] (see 1.4.1) calculates the Hochschild-Mitchell homology of $\mathbb{C}$, with coefficients in the bifunctor $\mathbb{C}(-,-): \mathbb{C}^{0} \times \mathbb{C} \rightarrow \mathscr{A} b$, that is $\pi_{k}(F .(\mathbb{C}))=\pi_{k}\left(C N(\mathbb{Z}, \mathbb{C})=H_{k}(\mathbb{C}, \mathbb{C})\right.$. In [12] it was shown, by means of universal properties, that $H_{k}\left(\mathscr{P}_{A}, \mathscr{P}_{A}\right)$ is isomorphic to $\pi_{k}(T H H(A)$. This may also be viewed as a immediate corollary of the result above.

### 2.2.5. Corollary. Let $\mathbb{C}$ be an additive category. Then

$$
H_{k}(\mathbb{C}, \mathbb{C}) \cong \pi_{k} T H H(\mathbb{C})
$$

In particular, if $A$ is an associative ring with unit, then

$$
H_{k}\left(\mathscr{P}_{A}, \mathscr{P}_{A}\right) \cong \pi_{k}(T H H(A))
$$

Proof. The second statement follows from the first by Morita equivalence of topological Hochschild homology.

Consider $\mathbb{C}$ as an exact category by choosing the split exact sequences. We have an inclusion $F .(-) \subset R(-)$. In particular $F_{0}(-) \rightarrow R_{0}(-)$ is a homotopy equivalence. Consider the commutative diagram


The left vertical maps are equivalences by [7] together with Lemma 2.2.2 and the others follow from results in this section.

It should again be stressed that these results are false if $\mathfrak{C}$ is only a linear category. Although the $S$ construction does not appear in the statement, it is the ability to
make sums within the category which provide the isomorphisms (or more constructively: it is the additivity of $\operatorname{THH}\left(S_{-}\right)$and $F\left(S_{-}\right)$which make the parallel reductions possible).
2.2.6. Generalizations of the above results. One immediate generalization of Corollary 2.2.4 is the following. If $X$ is some finite pointed simplicial set then tensoring with $\tilde{Z}[X]$ makes sense, and we may form the simplicial $S^{(k)} \mathbb{C}$ bimodule $S^{(k)} \mathbb{C}(-,-\otimes \tilde{Z}[X])$ sending $a, b \in S^{(k)} \mathbb{C}$ and $Y \in f s_{*} \mathscr{E} n s$ to $S^{(k)} \mathbb{C}(a, b \otimes \tilde{Z}[X \wedge Y])$. The proof of Lemma 2.2 .3 obviously extends to this case proving that

$$
\begin{aligned}
& \lim _{k \rightarrow \infty} \Omega^{k} T H H_{0}\left(S^{(k)} \mathbb{C}, S^{(k)} \mathbb{C}(-,-\otimes \tilde{Z}[X])\right) \\
& \quad \subseteq \lim _{k \rightarrow \infty} \Omega^{k} T H H\left(S^{(k)} \mathbb{C}, S^{(k)} \mathbb{C}(-,-\otimes \tilde{Z}[X])\right)
\end{aligned}
$$

is a homotopy equivalence, and the latter object is equivalent to $\Omega \operatorname{THH}(S \mathbb{C}, S \mathbb{C}(-,-\otimes \tilde{\boldsymbol{Z}}[X]))$.

Another thing one should remark is that the reduction of 1.4.7 did not require the ring functor itself to be linear. This may be exploited if one wishes to study the effect of various subcategories of weak equivalences on some particular linear bimodule, say of the type just mentioned. Here one should notice that the natural transformations defined in the proof of Proposition 2.2.3 will not take a category of weak equivalences outside itself. In fact, it was mainly as a preparation for such applications that the changes from [7] in the natural transformations were made. These questions will hopefully be treated in a later paper, and may prove crucial in an attempt to understand stable algebraic K-theory of spaces in this context.

### 2.3. Reduction by resolution

As an example of a translation from K-theory to the topological Hochschild homology using the description of $T H H$ in the previous section, we offer the following calculation. One should note that this approach is only desirable if one is satisfied with getting non-cyclic information. This may however be remedied by use of the principle of Goodwillie (see the discussion following 1.5.3), which roughly says that any natural property of the homotopy type of topological Hochschild homology extends to the fixed point sets of the finite actions. Thus, to prove a theorem of this sort about $\Omega T H H\left(S_{-}\right)$it is enough to prove it for $\Omega T H H_{0}(S-)$.
2.3.1. The exact category $\mathscr{E}_{X}(\mathbb{C})$. For any exact category $\mathbb{C}$ and finite pointed set $X$ let $\mathscr{E}_{X}(\mathbb{C})$ denote the exact category of pairs $(c, v)$ where $c \in \mathbb{C}$ and $v: c \rightarrow c \otimes \tilde{Z}[X]$ a map in $\mathbb{C}$. A morphism in $\mathscr{E}_{X}(\mathbb{C})$ form $(c, v)$ to $(d, w)$ is a morphism $f: c \rightarrow d$ in $\mathbb{C}$ such that

commutes. A sequence in $\mathscr{E}_{X}(\mathbb{C})$ is exact if and only if the associated sequence of objects in $\mathbb{C}$ is exact in $\mathbb{C} . \mathscr{E}_{\boldsymbol{X}}(\mathbb{C})$ is clearly functiorial in both $X$ and $\mathbb{C}$, and set maps resp. exact functors are sent to exact functors. Note in particular that $\mathbb{C}$ equals $\mathscr{E}_{\boldsymbol{*}}(\mathbb{C})$ and $\mathscr{E}_{S^{\circ}}(\mathbb{C})=\operatorname{End}(\mathbb{C})$. In fact if $X$ has $n+1$ elements, then $\mathscr{E}_{X}(\mathbb{C})$ is the category with objects $\left(c ;\left\{v_{i}\right\}\right)$ where $c \in \mathbb{C}$ and $v_{i} \in \operatorname{End}(c) i=1, \ldots, n$ and where a morphism $\left(c,\left\{v_{i}\right\}\right) \rightarrow\left(d,\left\{w_{i}\right\}\right)$ is a morphism $f: c \rightarrow d$ such that $f \circ v_{i}=w_{i} \circ f$ for all $i$. For a simplicial set we apply $\mathscr{E}_{-}(\mathbb{C})$ degreewise.
2.3.2. Proposition (The equivalence criterion). Let $F$ be an exact functor such that $S^{(k)}\left(\mathscr{E}_{X}(F)\right)$ is a weak equivalence for some $k>0$ and all finite sets $X$. Then $T H H(S F)$ is a weak equivalence.

Proof. Consider the $S$ construction on $\mathscr{E}_{X}(\mathbb{C})$ :

$$
S\left(\mathscr{E}_{X}(\mathbb{C})\right)=\coprod_{c \in S \mathbb{E}} S \mathbb{C}(c, c) \otimes \tilde{Z}[X]
$$

The homotopy colimit (over $n$ ) of the cofibre of the maps $S^{(k)} \mathbb{C}=S^{(k)} \mathscr{E}_{*}(\mathscr{C}) \subseteq$ $S^{(k)}\left(\mathscr{E}_{S^{n}}(\mathbb{C})\right)$ is thus exactly $T H H_{0}\left(S^{(k)}(\mathbb{C})\right.$, and so to prove a statement in $T H H(S-)$, by Proposition 2.2.3 it is enough to prove it for each of the $S^{(k)}\left(\mathscr{E}_{S^{n}}(\mathbb{C})\right)$ for $n \geq-1$.
2.3.3. Proposition (Resolution theorem for THH). Let © be a full subcategory of an exact category $\mathfrak{D}$ which is closed under extension. Assume that
(1) If $0 \rightarrow d^{\prime} \rightarrow c \rightarrow d^{\prime \prime} \rightarrow 0$ is an exact sequence in $\mathfrak{D}$ with $c \in \mathbb{C}$, then $d^{\prime} \in \mathbb{C}$.
(2) For any $d^{\prime \prime} \in \mathfrak{D}$ there exists such a short exact sequence with $c$ projective in $\mathfrak{D}$.

Then $\operatorname{THH}(S \mathbb{C}) \rightarrow \operatorname{THH}(S \mathfrak{D})$ is a weak equivalence.
Proof. Note that for the K-theory situation, the projectivity assumption in (2) is unnecessary. We need to show that, for all finite sets $X$, the categories $\mathscr{E}_{X}(\mathbb{C}) \subseteq \mathscr{E}_{X}(\mathbb{D})$ fulfill the requirements. It is clear that $\mathscr{E}_{X}(\mathbb{C})$ is a full subcategory of $\mathscr{E}_{X}(\mathcal{D})$. Furthermore, as a sequence

$$
0 \rightarrow\left(d^{\prime}, v^{\prime}\right) \rightarrow(d, v) \rightarrow\left(d^{\prime \prime}, v^{\prime \prime}\right) \rightarrow 0
$$

in $\mathscr{E}_{X}(\mathcal{D})$ is exact iff $0 \rightarrow d^{\prime} \rightarrow d \rightarrow d^{\prime \prime} \rightarrow 0$ is exact, and $(d, v) \in \mathscr{E}_{X}(\mathbb{C})$ iff $d \in \mathbb{C}(\mathbb{C}$ is full) it is clear that $\mathscr{E}_{X}(\mathbb{C})$ is closed under extensions and admissible subobjects (requirement 1). As to requirement 2 , let $\left(d^{\prime \prime}, v^{\prime \prime}\right) \in \mathscr{E}_{x}(\mathfrak{D})$. Choose an exact sequence

$$
0 \rightarrow d^{\prime} \rightarrow c \rightarrow d^{\prime \prime} \rightarrow 0
$$

with $c \in \mathbb{C}$. As $c$ is projective there exists a lifting $v: c \rightarrow c \otimes \tilde{Z}[X]$ in


This gives an exact sequence

$$
0 \rightarrow\left(d^{\prime}, v \mid d^{\prime}\right) \rightarrow(c, v) \rightarrow\left(d^{\prime \prime}, v^{\prime \prime}\right) \rightarrow 0
$$

and we are done.

Hence, for instance, if $A$ is a regular ring, and $\mathfrak{D}$ is the category of finitely presented modules, we have that the map $\operatorname{THH}(A) \rightarrow \Omega T H H(S D)$ induced by the inclusion $\mathscr{P}_{A} \rightarrow \mathfrak{D}$ is a weak equivalence.

The resolution theorem also provides us with a simple example showing that the splitness assumption in Proposition 2.1.3 was necessary.
2.3.4. An example where $T H H(\mathbb{C}) \nsim \Omega T H H(S \mathbb{C})$. Let $\mathbb{C}$ be the category of finitely generated abelian groups with all exact sequences. By the resolution theorem we have

$$
\Omega T H H(S \mathbb{C}) \simeq \Omega T H H\left(S \mathscr{F}_{z}\right) \simeq T H H\left(\mathscr{F}_{z}\right) \simeq T H H(Z)
$$

where for any ring $A \mathscr{F}_{A}$ is the split exact category of finitely generated $A$-modules. The equivalence in the middle is due to Proposition 2.1.3, whereas the last is clear from the proof of Proposition 2.1.5. For any prime p, consider the linear (not exact) functor $\operatorname{Tor}_{1}^{\boldsymbol{Z}}(-, \boldsymbol{Z} / p \boldsymbol{Z}): \mathbb{C} \rightarrow \mathscr{F}_{\boldsymbol{Z} / p \boldsymbol{Z}}$. This is split by the inclusion $\mathscr{F}_{\boldsymbol{Z} / p \boldsymbol{Z}} \subseteq \mathbb{C}$, so

$$
T H H(\mathbb{C}) \simeq T H H\left(\mathscr{F}_{z / p z}\right) \times ?
$$

But by the calculations of Bökstedt, THH $\left(\mathscr{F}_{Z / p Z}\right) \simeq T H H(Z / p Z)$ is not a factor of THH(Z):

$$
\begin{aligned}
\pi_{\mathrm{k}} T H H(Z / p Z) & = \begin{cases}\boldsymbol{Z} / p Z, & \text { if } k=2 i \geq 0 \\
0, & \text { otherwise }\end{cases} \\
\nsubseteq \pi_{k} T H H(Z) & = \begin{cases}Z, & \text { if } k=0, \\
Z / i Z, & \text { if } k=2 i-1 \geq 0, \\
0, & \text { otherwise }\end{cases}
\end{aligned}
$$

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